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New Robust Stability and Stabilization Conditions for Linear Repetitive Processes

Wojciech Paszke
W.Paszke@tue.nl
Olivier Bachelier
Olivier.Bachelier@univ-poitiers.fr

Technische Universiteit
Eindhoven
University of Technology

Where innovation starts

Outline

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- Linear Repetitive Processes
  - Stability Theory for Linear repetitive processes
  - Stability along the pass
- Robust stability
- Robust stabilization
- Numerical Example
- Conclusions and future works

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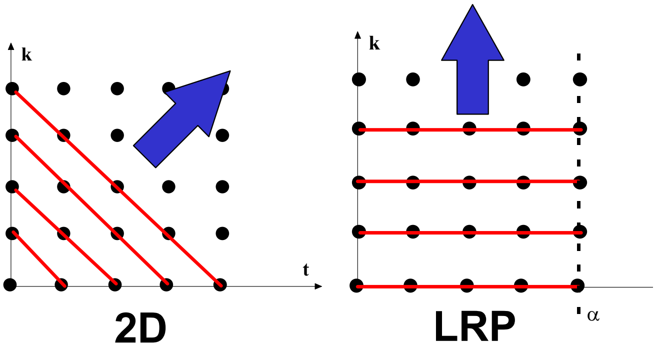
Linear Repetitive Processes — Basics

A repetitive process is defined by a series of sweeps, termed **passes,** through a set of dynamics defined over a finite and fixed duration known as **the pass length.** On each pass an output, termed **the pass profile,** is produced which acts as a forcing function on, and hence contributes to, the dynamics of the next pass profile.

Key properties

- **repetitive operation**
  each individual execution of operation is termed "pass",
- **finite pass length**
- **interpass interaction**
  interaction between the state and/or output functions generated during successive execution of operation.

Repetitive Processes and 2-D Systems



- Strong structural links but only in some cases.
- Repetitive processes have dynamics which have no counterparts in other classes of 2-D systems.

Applications

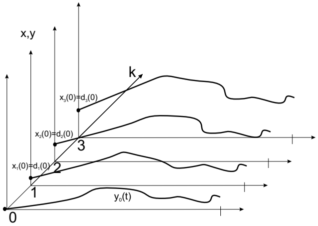
Physical examples

- Metal rolling operations.
- Long-wall coal cutting.
- Hard disk drives.
- Spatially-distributed systems.
- Distillation Column modelling.
- Vehicle Convoys.

Algorithmic examples

- Iterative learning control (ILC).
- Iterative algorithms for solving nonlinear dynamic optimal control problems based on the maximum principle.

Differential LRPs



- The state space model (over $0 \leq t \leq \alpha$, $k \geq 0$)

$$\begin{aligned} \dot{x}_{k+1}(t) =& A x_{k+1}(t) + B_0 y_k(t) + B u_{k+1}(t) \\ y_{k+1}(t) =& C x_{k+1}(t) + D_0 y_k(t) + D u_{k+1}(t) \end{aligned}$$

- Boundary conditions
  - the state initial vector on each pass $x_{k+1}(0)$, $k \geq 0$,
  - the initial pass profile $y_0(t) = f(t)$.

LRPs – discrete time model

State space model

$$
\begin{aligned}
x_{k+1}(p+1) &= Ax_{k+1}(p) + B_0 y_k(p) + Bu_{k+1}(p) \\
y_{k+1}(p) &= Cx_{k+1}(p) + D_0 y_k(p) + Du_{k+1}(p)
\end{aligned}
$$

where

- $p = \{0, 1, \cdots, \alpha - 1\}$
- $A, \ B, \ B_0, \ C, \ D, \ D_0$ – matrices computed from differential model by the particular **discretization method** used with the discretization period T.

Now the boundary conditions are

$$
\begin{aligned}
x_{k+1}(0) &= d_{k+1}, \ k = 0, 1, \ldots \\
y_0(p) &= f(p), \ p = 0, 1, \ldots, \alpha - 1.
\end{aligned}
$$

Stability theory

This consists of two distinct concepts

- **asymptotic stability** - basically BIBO stability over the pass length.
- **stability along the pass** - basically BIBO stability uniformly, i.e. independent of the pass length.

Lemma

Linear Repetitive Process is asymptotically stable iff

$$\rho(D_0) < 1$$

Asymptotic stability $\Rightarrow$ *that* $\{y_k\}_{k\geq 1}$ *converges as* $k \to \infty$ *to the limit profile*

$$\begin{aligned} x_\infty(p+1) &= \left(A + B_0(I - D_0)^{-1}C\right) x_\infty(p) + Bu_\infty(p) \\ y_\infty(p) &= (I - D_0)^{-1} C x_\infty(p), \quad x_\infty(0) = d_\infty \end{aligned}$$

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A somewhat surprising result — independent of the state updating dynamics!!

Resulting limit profile

$$
\begin{aligned}
x_\infty(p+1) &= \left(A + B_0(I - D_0)^{-1}C\right)x_\infty(p) + Bu_\infty(p)\\
y_\infty(p) &= (I - D_0)^{-1}Cx_\infty(p), \quad x_\infty(0) = d_\infty
\end{aligned}
$$

This is a standard or 1-D linear system — ‘nice’(?) in terms of control or not?

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In fact, asymptotic stability does not guarantee 'acceptable' limit profile dynamics.

Example

$$
\begin{aligned}
x_{k+1}(p+1) =& -0.5x_{k+1}(p) + u_{k+1}(p) + (0.5+\beta)y_k(p) \\
y_{k+1}(p) =& x_{k+1}(p),\ x_{k+1}(0) = 0,\ 0 \le p \le \alpha - 1
\end{aligned}
$$

This process is asymptotically stable but the resulting limit profile over $0 \le p \le \alpha - 1$

$$y_\infty(p+1) = \beta y_\infty(p) + u_\infty(p)$$

is unstable in 1D sense if $|\beta| \ge 1$. ('Growth' term in the the along the pass direction.)

Stability along the pass

The problem here is the finite pass length. Stability along the pass considers the case as $\alpha \to \infty$

Theorem

Suppose that the pair $\{A, B_0\}$ *is controllable and the pair* $\{C, A\}$ *is observable. Then the discrete linear repetitive process is stable along the pass if, and only if,*

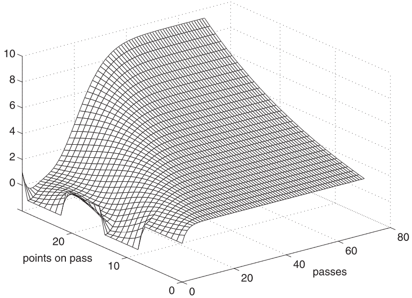
1. $\rho(D_0) < 1$
2. $\rho(A) < 1$
3. *all eigenvalues of* $G(z) = C(zI - A)^{-1}B_0 + D_0$ *have modulus strictly less than unity* $\forall |z| = 1$

Stability along the pass

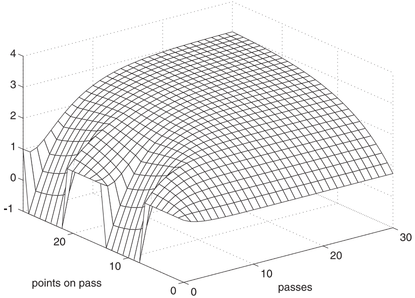
Note for the simple example above $r(A) < 1$ is only a necessary condition — we also need the third condition (physical meaning (differential case) each frequency component of the initial pass profile must be attenuated from pass-to-pass and not just d.c. content (asymptotic stability)).

For dynamic boundary conditions — asymptotic stability condition becomes much more complex. Possible to test in discrete case by a 1-D equivalent model — this is not the same as just applying 1-D linear systems theory — in the differential case – still a headache!!!

Stability along the pass



Unstable along the pass process



Stable along the pass process

Lemma

Assume that there exist matrices $P_1 \succ 0$ and $P_2 \succ 0$ of compatible dimensions such that LMI

$$\begin{bmatrix} -P_2 & P_2 C & P_2 D_0 \\ C^T P_2 & A^T P_1 + P_1 A & P_1 B_0 \\ D_0^T P_2 & B_0^T P_1 & -P_2 \end{bmatrix} \prec 0$$

or

$$\begin{bmatrix} -P_1 & 0 & P_1 A & P_1 B_0 \\ 0 & -P_2 & P_2 C & P_2 D_0 \\ A^T P_1 & C^T P_2 & -P_1 & 0 \\ B_0^T P_1 & D_0^T P_2 & 0 & -P_2 \end{bmatrix} \prec 0$$

hold, then the differential LRP (resp. the discrete LRP) is stable along the pass.

Towards a better result

- the Lyapunov matrices (P_1 and P_2) are not separated from the process matrices;
- there are no slack matrix variables which can introduce an additional flexibility in obtaining a solution;

Notation

- δ stands for the derivation or shifting operator (depending on the differential or purely discrete case)
- $$\mathcal{A} = \begin{bmatrix} A & B_0 \\ C & D_0 \end{bmatrix} \quad \text{and} \quad \mathcal{B} = \begin{bmatrix} B \\ D \end{bmatrix}$$
- $$\mathrm{sym}(X) = X + X^T$$

Stability along the pass in term of LMI

Theorem

Assume that there exist matrices $Y_1 \succ 0$, $Y_2 \succ 0$ and G of compatible dimensions such that

$$\Upsilon + \mathrm{sym}\left(\begin{bmatrix} \mathcal{A} \\ -I \end{bmatrix} G \mathscr{I}\right) \prec 0$$

where

$$\Upsilon = \begin{bmatrix} 0 & 0 & Y_1 & 0 \\ 0 & -Y_2 & 0 & 0 \\ Y_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & Y_2 \end{bmatrix} \quad \text{or} \quad \Upsilon = \begin{bmatrix} -Y_1 & 0 & 0 & 0 \\ 0 & -Y_2 & 0 & 0 \\ 0 & 0 & Y_1 & 0 \\ 0 & 0 & 0 & Y_2 \end{bmatrix}$$

$$\mathscr{I} = \begin{bmatrix} I & 0 & I & 0 \\ 0 & 0 & 0 & I \end{bmatrix} \quad \text{or} \quad \mathscr{I} = \begin{bmatrix} 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{bmatrix}$$

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Standard vs. Recent results

1. Improved results are provided without introducing any additional degree of conservativeness

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1. Improved results are provided without introducing any additional degree of conservativeness
2. Improved results are obtained by mean of the matrix elimination procedure
3. The matrix slack variable G has full form (not block diagonal) and introduces extra degree of freedom
4. The matrix G is particularly useful in the robust context to introduce parameter-dependent Lyapunov functions.

A static feedback controller case

The control law

$$u_{k+1}(p) = \begin{bmatrix} K_1 & K_2 \end{bmatrix} \begin{bmatrix} x_{k+1}(p) \\ y_k(p) \end{bmatrix}$$

where

- *K_1 and K_2 are matrices to be designed.*

then the closed-loop process is

$$\delta \xi_k = \mathcal{A}_c \xi_k$$

where

$$\mathcal{A}_c = \mathcal{A} + \mathcal{B} K$$

Stabilization of nominal LRP

Theorem

Suppose that an LRP is subjected to the static control law then the resulting closed-loop process is stable along the pass if there exist matrices $Y_1 \succ 0$, $Y_2 \succ 0$, G and L such that

$$\Upsilon + \mathrm{sym}\left(\left(\begin{bmatrix} \mathcal{A} \\ -I \end{bmatrix} G + \begin{bmatrix} \mathcal{B} \\ 0 \end{bmatrix} L\right) \mathcal{I}\right) \prec 0$$

In this case, the controller matrix is given by

$$K = LG^{-1}$$

Robust stability

Remarks:

- unlike previous results where matrices K_1 and K_2 are directly deduced from dual Lyapunov matrices Y_1 and Y_2 , matrix K is here computed with L and G
- this can bring additional flexibility, reducing conservativeness, especially for the uncertain process case
- matrix G allows us to introduce parameter-dependent Lyapunov functions when the model is itself parameter-dependent

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Indeed, assume that a process is actually subject to uncertainty such that it can be written

$$\delta\xi_{k+1} = \mathbb{A}\xi_k + \mathbb{B}u_{k+1}$$

where

$$\begin{bmatrix} \mathbb{A} & \mathbb{B} \end{bmatrix} = \begin{bmatrix} \mathcal{A} & \mathcal{B} \end{bmatrix} + \begin{bmatrix} \Delta_A & \Delta_B \end{bmatrix}.$$

Uncertainty structure

Furthermore, it is assumed that matrices $\mathcal{A}$ and $\mathcal{B}$ dependent on a real parameter vector θ with a polytopic dependency:

$$\begin{bmatrix} \mathcal{A} & \mathcal{B} \end{bmatrix} = \begin{bmatrix} \mathcal{A}(\theta) & \mathcal{B}(\theta) \end{bmatrix} = \sum_{i=1}^{N} \theta_i \begin{bmatrix} \mathcal{A}_i & \mathcal{B}_i \end{bmatrix}, \quad \theta \in \Theta$$

where

$$\Theta = \left\{ \theta = \begin{bmatrix} \theta_1 \\ \vdots \\ \theta_N \end{bmatrix} : \theta_i \geq 0, \sum_{i=1}^{N} \theta_1 = 1 \right\}.$$

It means that the various matrices $[\mathcal{A}_i \ \ \mathcal{B}_i]$, which are known, are the vertices of a polytope in which the actual matrix $[\mathcal{A} \ \ \mathcal{B}]$ lies.

Uncertainty structure

Moreover, the additive uncertainties are assumed to be

$$\begin{cases} \Delta_A = H_A \mathcal{F}_A E_A, & \|\mathcal{F}_A\|_2 \leq \rho_A, \\ \Delta_B = H_B \mathcal{F}_B E_B, & \|\mathcal{F}_B\|_2 \leq \rho_B. \end{cases}$$

- H_A, H_B, E_A and E_B give some desired structure to the additive uncertainty
- $\mathcal{F}_A$ and $\mathcal{F}_B$ are uncertain matrices that belong to some balls of matrices whose respective radii are ρ_A and ρ_B.

To make this description even more general, the matrices H_A, H_B, E_A and E_B are also subjected to a polytopic dependency:

$$\begin{aligned} \begin{bmatrix} H_A & H_B & E_A^T & E_B^T \end{bmatrix} &= \begin{bmatrix} H_A(\theta) & H_B(\theta) & E_A^T(\theta) & E_B^T(\theta) \end{bmatrix} \\ &= \sum_{i=1}^{N} \theta_i \begin{bmatrix} H_{A_i} & H_{B_i} & E_{A_i}^T & E_{B_i}^T \end{bmatrix}, \ \theta \in \Theta. \end{aligned}$$

Robust stabilization

In this case, the uncertain closed-loop model becomes

$$\delta\xi_k = (\mathcal{A}_c + \Delta_A + \Delta_B K)\xi_k = \mathbb{A}_c(\theta)\xi_k,$$

where $\mathcal{A}_c = \mathcal{A} + \mathcal{B}K$ inherits from the polytopic structure of $\mathcal{A}$ and $\mathcal{B}$. It is however false to claim that $\mathbb{A}_c$ is polytopic.

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Definition

The uncertain closed-loop repetitive process is robustly stable along the pass if and only if it is stable along a pass for any value of θ in Θ and for any matrices $\mathcal{F}_A$ and $\mathcal{F}_B$ such that $||\mathcal{F}_A||_2 \leq \rho_A$, $||\mathcal{F}_B||_2 \leq \rho_B$.

Robust stabilization

Theorem

Suppose that an LRP is subjected to the static control law then the resulting closed-loop process is stable along if there exist matrices $Y_{1_i} \succ 0$, $Y_{2_i} \succ 0$, $i = 1, \ldots, N$, as well as matrices G and L such that

$$M_i = \left[\begin{array}{c|c|c} Q_i & \ell^T \begin{bmatrix} G^T E_{A_i}^T & 0 \\ 0 & L^T E_{B_i}^T \end{bmatrix} & \begin{bmatrix} H_{A_i} & H_{B_i} \\ 0 & 0 \end{bmatrix} \\ \hline (\star) & \begin{bmatrix} -\rho_A^{-1} & 0 \\ 0 & -\rho_B^{-1} \end{bmatrix} & 0 \\ \hline (\star) & (\star) & \begin{bmatrix} -\rho_A^{-1} & 0 \\ 0 & -\rho_B^{-1} \end{bmatrix} \end{array}\right] \prec 0$$

$$\forall i \in \{1, \ldots, N\}$$

where

$$Q_i = \Upsilon_i + \operatorname{sym}\left(\begin{bmatrix} \mathcal{A}_i G + \mathcal{B}_i L \\ -I \end{bmatrix} \ell\right)$$

with Υ_i defined by indexing Y_1 and Y_2 in Υ. In this case, the matrix K is given by $K = LG^{-1}$.

Robust stabilization

Here, it has to be noticed that dual Lyapunov matrices involved in Υ and thus in Q also depend on θ in a polytopic way, i.e.

$$\begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} = \begin{bmatrix} Y_1(\theta) \\ Y_2(\theta) \end{bmatrix} = \sum_{i=1}^{N} \theta_i \begin{bmatrix} Y_{1_i} \\ Y_{2_i} \end{bmatrix}, \quad \theta \in \Theta$$

Finally, the convex programming problem that consists in minimizing the objective function can be solved

$$\mathcal{J} = \alpha_A \rho_A^{-1} + \alpha_B \rho_B^{-1}$$

Numerical example (part1/3)

Consider the differential LRP represented by

$$\left[\,\mathcal{A}_1 \;\middle|\; \mathcal{B}_1\,\right] = \left[\begin{array}{ccc|c} 0.1086 & 0.1538 & 0.0908 & 1.7362 \\ 0.0601 & 0.0694 & 0.0823 & 0.3614 \\ 0.0598 & 0.0084 & 0.1438 & 1.9599 \end{array}\right],$$

$$\left[\,\mathcal{A}_2 \;\middle|\; \mathcal{B}_2\,\right] = \left[\begin{array}{ccc|c} 1.0860 & 1.5380 & 0.9084 & 1.9292 \\ 0.6008 & 0.6944 & 0.8235 & 0.4016 \\ 0.5982 & 0.0837 & 1.4385 & 2.1777 \end{array}\right],$$

$$\left[\,\mathcal{A}_3 \;\middle|\; \mathcal{B}_3\,\right] = \left[\begin{array}{ccc|c} 1.3032 & 1.8457 & 1.0901 & 2.3150 \\ 0.7210 & 0.8333 & 0.9881 & 0.4819 \\ 0.7178 & 0.1004 & 1.7261 & 2.6132 \end{array}\right]$$

Numerical example (part2/3)

Norm-boud uncertainty is modelled with

$$\left[\, H_{A_1} \mid H_{A_2} \mid H_{A_3} \,\right] = \left[\begin{array}{c|c|c} 0.0757 & 0.0773 & 0.0287 \\ 0.0609 & 0.0573 & 0.0752 \\ 0.0967 & 0.0450 & 0.0094 \end{array}\right],$$

$$\left[\, H_{B_1} \mid H_{B_2} \mid H_{B_3} \,\right] = \left[\begin{array}{c|c|c} 0.0107 & 0.0581 & 0.0360 \\ 0.0735 & 0.0299 & 0.0718 \\ 0.0355 & 0.0714 & 0.0995 \end{array}\right],$$

$$\begin{aligned} E_{A_1} &= \begin{bmatrix} 0.0041 & 0.0329 & 0.0709 \end{bmatrix}, \quad E_{B_1} = 0.0161 \\ E_{A_2} &= \begin{bmatrix} 0.0860 & 0.0501 & 0.0995 \end{bmatrix}, \quad E_{B_2} = 0.0159 \\ E_{A_3} &= \begin{bmatrix} 0.0429 & 0.0267 & 0.0473 \end{bmatrix}, \quad E_{B_3} = 0.0268. \end{aligned}$$

Numerical example (part3/3)

Design procedure for $\rho_A = 0.1$ and $\rho_B = 0.1$ gives the solution as

$$G = \begin{bmatrix} -397.2800 & -159.6062 & 10.8490 \\ -560.0837 & -237.1113 & -2.5830 \\ 198.1517 & 79.5747 & -4.1740 \end{bmatrix},$$

$$L = \begin{bmatrix} -0.1723 & 0.2542 & -0.1162 \end{bmatrix}$$

and the corresponding controller matrix is

$$K = \left[\begin{array}{cc|c} -0.1961 & -0.0255 & -0.4661 \end{array}\right].$$

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Conclusions

- New results on the relatively open problem of robust control of linear repetitive processes have been developed
- Separation of the Lyapunov matrices from the process matrices has been presented - LMI conditions based on parameter dependent Lyapunov functions can be derived
- The obtained result can be extended to the case of a quite general affecting uncertainty affecting the model, namely the polytopic normbounded uncertainty.
- Uncertainty is present on both the state dynamics and the pass profile updating equations of the state space model.