Empirical Mode Decomposition for vectorial bi-dimensional signals

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6th International Workshop on Multidimensional (nD) Systems

Outlines

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Empirical Modale Decomposition?

- Idea– any signal $x(t)$ can be seen as the superposition of many rapid and slow oscillations (*Huang and al*)
- Purposes– Extract this oscillations by decomposing:

$$x(t) = \sum_k c_k(t) + r(t),$$

  $c_k(t)$ intrinsic modes functions (IMFs) and $r(t)$ is a tendency
- The IMFs– are function verifying:
  1. *Local symmetry*, they have a vanishing local mean
  2. oscillations: the maxima (resp. minima) are strictly positives (resp. negatives)

Why doing an EMD?

- Avoid the limits of the usual time-frequency analysis Fourier and Wavelets, *Huang and al 1998*
- More suitable for non stationary and non linear systems
- It has the advantage to not use an a priori bases, so more freedom.
- Finally in term in Hilbert transform and the notion of instantaneous frequencies

$$c_k(t) = Re\left\{a_k(t)\exp\{\jmath\int 2\pi f_k(t)dt\}\right\}, \; f_k(t) \text{ make sense}$$

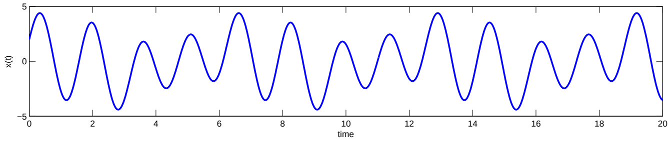
$$x(t) = Re\left\{\sum_k a_k(t)\exp\{\jmath\int 2\pi f_k(t)dt\}\right\}$$

How to implement the EMD?

The sifting algorithm:

1. Find local extrema of $x(t)$.
2. Calculate the upper enveloppe $M(t)$ and the lower envelope $m(t)$ (using a cubic splines).
3. Update the signal, $x(t) \leftarrow x(t) - \frac{M(t)+m(t)}{2}$.
4. Repeat 1, 2 et 3 until having an IMF $c(t)$.
5. Substrat the IMF obtained in 4, $x(t) \leftarrow x(t) - c(t)$.
6. Repeat 1-5 until having a tendency $r(t)$ (a curve having at most one extremum)

Illustration of the sifting algorithm



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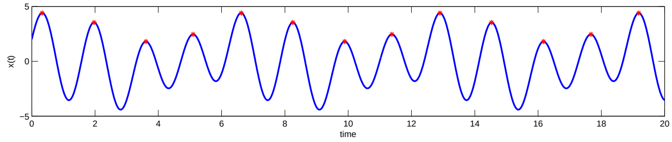

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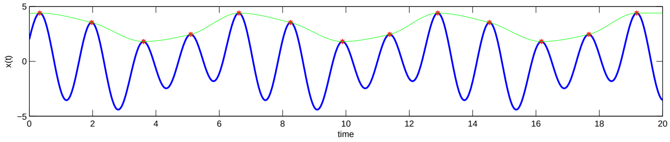

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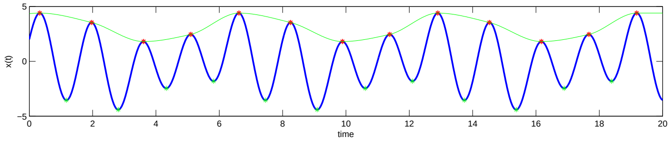

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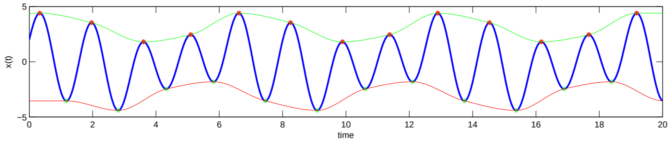

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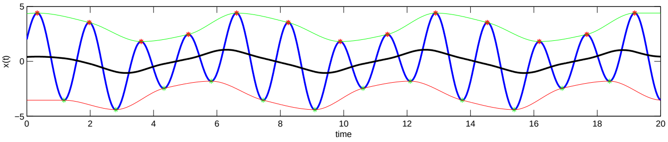

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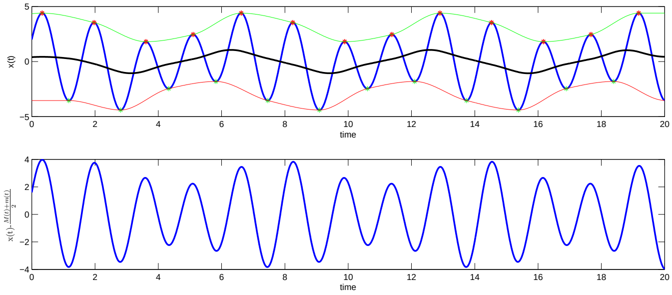

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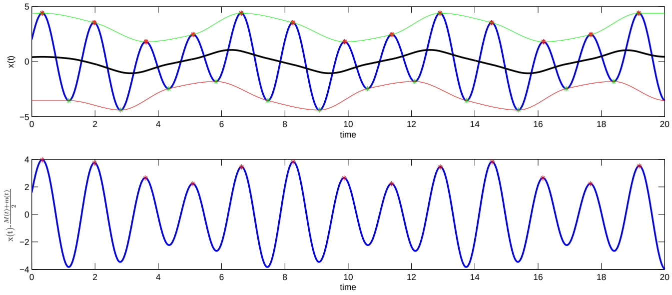

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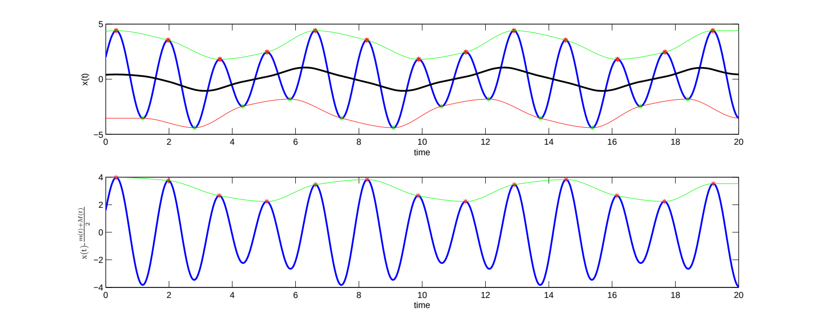

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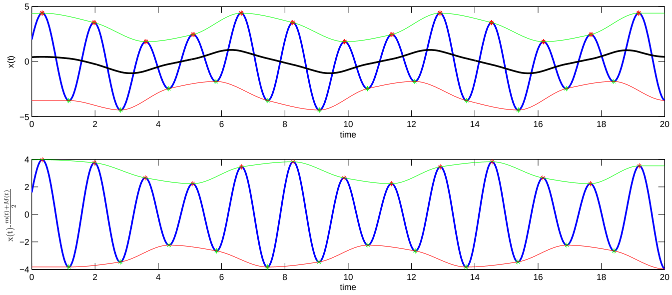

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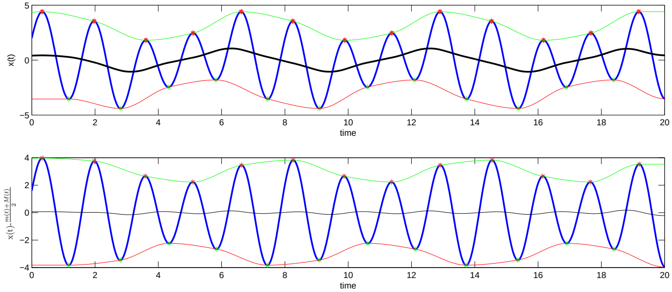


Figure: and so on...

Limitations of the EMD

Although EMD has a great success in applications it has some limitations:

- Problems linked to the numerical treatments
  - Determination of a stoping test for the sifting algorithm
  - Instability of the algorithm
  - Sensitivity to perturbations and sampling
- Problems linked to the conception of the EMD
  - Ambiguous definition of IMFs; local symmetry
  - No theoretical formulation
  - Not easy to generalize to vectorial signals (*Flandrin and al 2007*)

Outlines

Elementary Intrinsic Mode Functions (EIMF): the 1D case

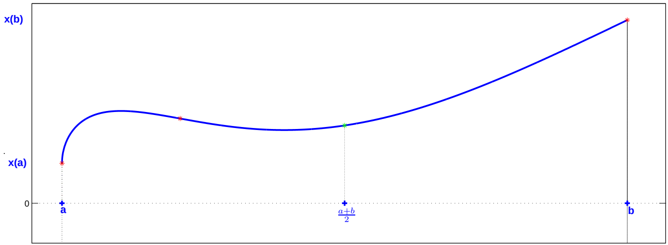
A regular function $x(t)$ is an EIMF if it verifies:

1. The function $x(t)$ has many inflexion points which are also zero-crossing (except eventually the end points of the signal).
2. For every three consecutive inflexions, the curve of $x(t)$ is symmetric with respect to the central point

$\Longrightarrow$ An EIMF is an IMF in Huang's sense.

- A tendency is a function having at most one inflexion point

A simple illustration of our idea in 1D case



The idea is to search for a function $T(t)$ such that $x(t) - T(t) = c(t)$ be an EIMF on $[a, b]$ and it verifies the central symmetry with respect to $(\frac{a+b}{2}, 0)$.

A first local formulation

Lemma

Let $x(t)$ a regular function having tow different type of convexity on $[a, b]$. Then it may be decomposed as $x(t) = c(t) + T(t)$ and,

$$c(t) = \frac{x(t) - x(a+b-t)}{2} - \varphi(2t - (a+b)),$$

where φ is any odd function linked to T by the equality:

$$\varphi(2t - (a+b)) = \frac{T(t) - T(a+b-t)}{2}$$

$\Longrightarrow$ There exist many couples (φ, T) which are solution of a such problem.

Uniqueness of the decomposition

Lemma

Let $x(t)$ be a signal as in last lemma, then the decomposition $x(t) = c(t) + T(t)$ is unique in the sense that:
φ is the only one such that the corresponding T did not change its convexity type (only convexe or only concave) on $[a, b]$.

- The fact that T do not change the convexity type on $[a, b]$ implies a less oscillation with respect to the extracted EIMF $c(t)$ (less inflexion points)
- This is the key idea of our algorithms convergence

Inline extraction in a general signal

Theorem

Let $x(t)$ be a regular function on $[A, B]$ and having $m = 2\ell - 1$ inflexion points $I_1(a_1, x(a_1)), \ldots, I_m(a_m, x(a_m))$. we denote $I_0(A, x(A))$ and $I_{m+1}(B, x(B))$. In this cas there exist , a set of odd functions $(\varphi_1, \ldots, \varphi_\ell)$ such that $x(t) = c(t) + T(t)$ where

$$c(t) = \sum_{i=1}^{\ell} \frac{x(t) - x(a_{2i-1} + a_{2i+1} - t)}{2} - \varphi_i(2t - (a_{2i-1} + a_{2i+1}))$$

and T is a function having at most ℓ inflexion points.
$\Longrightarrow$ *the convergence But how to determine the φ_i's ?*

A simple method: interpolation like

- The purpose is to approach φ_i on $[a_{2i-1}, a_{2i+1}]$ by a polynomial,

$$\varphi_i(t) = \alpha_i(t-m_i)+\beta_i(t-m_i)^3+\gamma_i(t-m_i)^5,\ m_i = \frac{a_{2i-1}+a_{2i+1}}{2}$$

- Let us denote $h_i = \frac{a_{2i+1}-a_{2i-1}}{2}$ and $D_i = \frac{x(a_{2i+1})-x(a_{2i-1})}{a_{2i+1}-a_{2i-1}}$, we will have:

$$\text{Porp. EIMF:} \qquad \alpha_i + \beta_i(h_i)^2 + \gamma_i(h_i)^4 = D_i$$

- Continuity of derivatives in a_{2i+1} gives the equations:

$$\varphi' : \alpha_{i+1} + 3\beta_{i+1}(h_{i+1})^2 + 5\gamma_{i+1}(h_{i+1})^4 = \alpha_i + 3\beta_i(h_i)^2 + 5\gamma_i(h_i)^4$$
$$\varphi'' : 3\beta_{i+1}(h_{i+1}) + 10\gamma_{i+1}(h_{i+1})^3 = 3\beta_i(h_i) + 10\gamma_i(h_i)^3$$

Advanced method: piecewise polynomial

- On a local interval $[a, b]$ we observe $x(t)$ at instants $t_0, t_1, \ldots, t_n, t_{n+1} = \frac{a+b}{2}$ et $a+b-t_n, a+b-t_{n-1}, \ldots, a+b-t_0 = b$
- in the left side, between successive instants $[t_i, t_{i+1}]$ we define the polynomials,

$$S_i^{\ell}(t) = \alpha_0^i + \alpha_1^i(2t-(a+b)) + \alpha_2^i(2t-(a+b))^2 + \alpha_3^i(2t-(a+b))^3,$$

- In the right side $[a+b-t_{i+1}, a+b-t_i]$,

$$S_i^{r}(t) = \beta_0^i + \beta_1^i(2t-(a+b)) + \beta_2^i(2t-(a+b))^2 + \beta_3^i(2t-(a+b))^3$$

$\Longrightarrow$ The purpose is to find α, β.

Advanced method: piecewise polynomials

- Using the fact that for $t = (t_i, i = 1, \dots, n)$

$$S_i^{\ell}(t) + S_i^{r}(a + b - t) = x(t) + x(a + b - t)$$

we have:

$$A_0^i + A_1^i(2t - (a + b)) + A_2^i(2t - (a + b))^2 + A_3^i(2t - (a + b))^3 = M(t)$$

where $M(t) = x(t) + x(a + b - t)$ and

$$\begin{cases} A_0^i = \alpha_0^i + \beta_0^i & ; \quad A_2^i = \alpha_2^i + \beta_2^i \\ A_1^i = \alpha_1^i - \beta_1^i & ; \quad A_3^i = \alpha_3^i - \beta_3^i \end{cases}$$

- The coefficients A_0^i, A_1^i, A_2^i and A_3^i are determined using a similar technic as in cubic splines

Advanced method: piecewise polynomial

- By symmetry, $\alpha_0^i = \beta_0^i = \dfrac{A_0^i}{2}$; $\alpha_2^i = \beta_2^i = \dfrac{A_2^i}{2}$
- We denote $X_i = 2t_i - (a+b)$ and compute,

$$B_{i+1} = \frac{(A_2^i - A_2^{i+1})}{4}(X_{i+1}) - \frac{(A_0^i - A_0^{i+1})}{4(X_{i+1})}$$

- we deduce the coefficients α, β (using continuity and derivatives),

$$\begin{aligned} \alpha_1^i &= \alpha_1^0 + \sum_{j=1}^{i} A_j \quad ; \quad \alpha_3^i = \alpha_3^0 + \sum_{j=1}^{i} B_j \\ \beta_1^i &= \alpha_1^i - A_1^i \quad ; \quad \beta_3^i = \alpha_3^i - A_3^i \end{aligned}$$

Advanced method: piecewise polynomial

- We compute

$$\mathbb{M}_1 = \min_i \left(A_2^i + \frac{\beta_2^i}{3(X_{i+1})} - \sum_{j=1}^{i} B_j \right); \mathbb{M}_2 = \max_i \left(\frac{-\alpha_2^i}{3(X_{i+1})} - \sum_{j=1}^{i} B_j \right)$$
$$\mathbb{M}_3 = \max_i \left(A_2^i + \frac{\beta_2^i}{3(X_{i+1})} - \sum_{j=1}^{i} B_j \right); \mathbb{M}_4 = \min_i \left(\frac{-\alpha_2^i}{3(X_{i+1})} - \sum_{j=1}^{i} B_j \right)$$

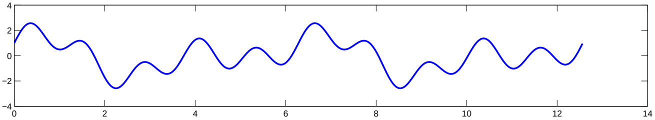
- if $x(\frac{a+b}{2}) \leq \frac{x(a)+x(b)}{2}$ (convexity) we take,

$$\mathbb{M}_1 \leq \alpha_0^3 \leq \mathbb{M}_2$$

- $x(\frac{a+b}{2}) \leq \frac{x(a)+x(b)}{2}$ (concavity) we take

$$\mathbb{M}_3 \leq \alpha_0^3 \leq \mathbb{M}_4$$

Illustration of the algorithm in action



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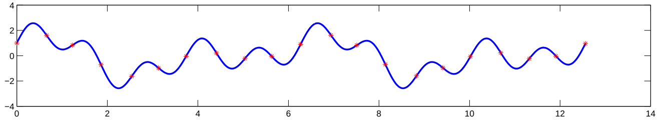

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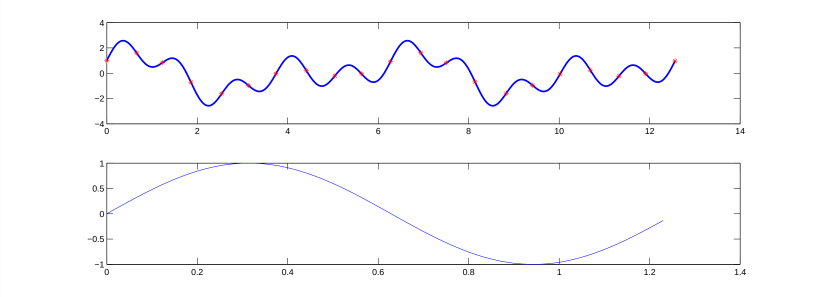

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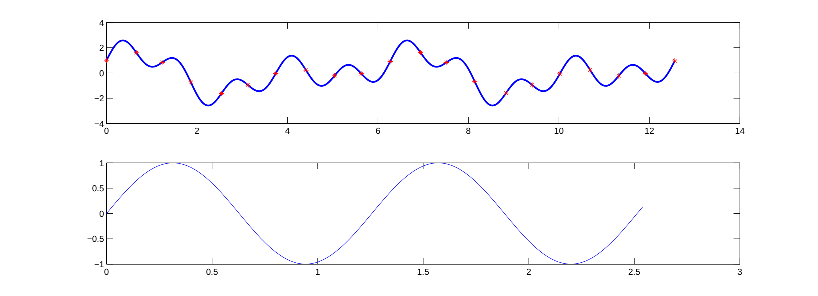

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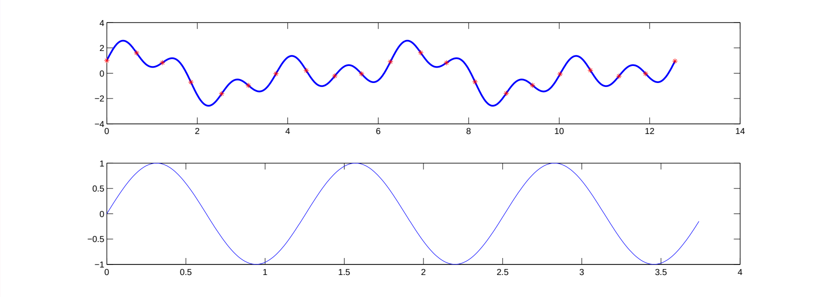

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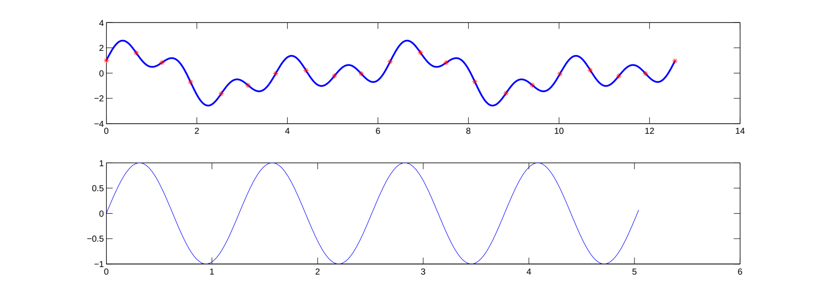

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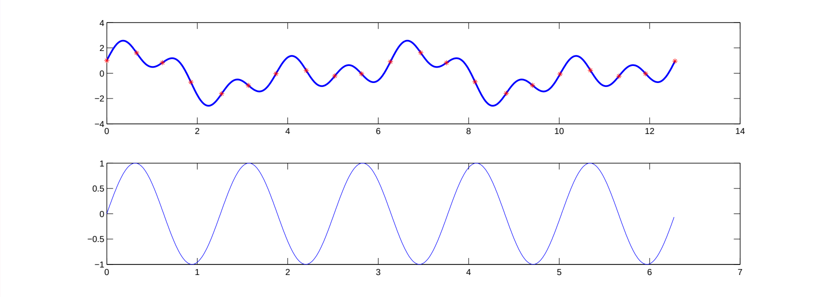

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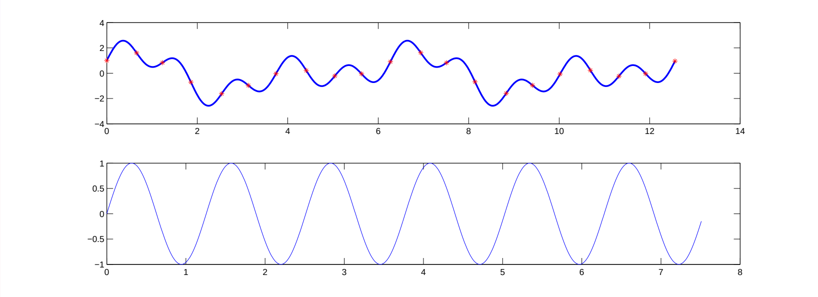

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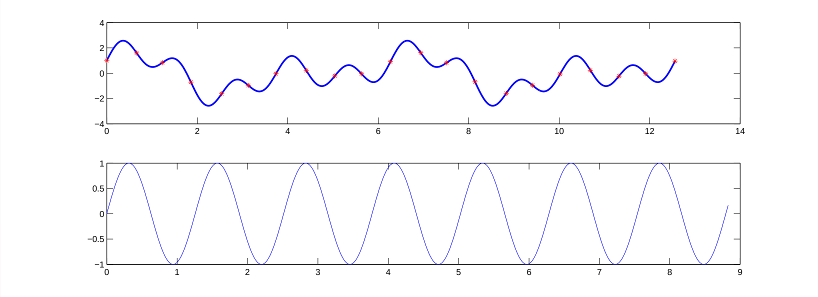

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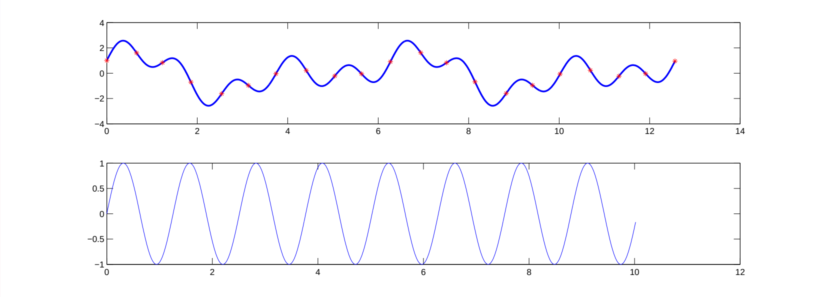

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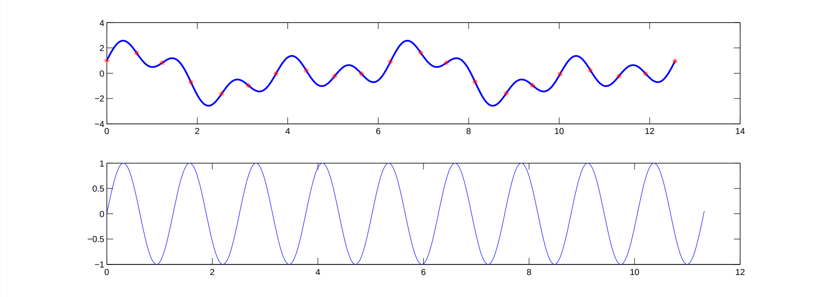

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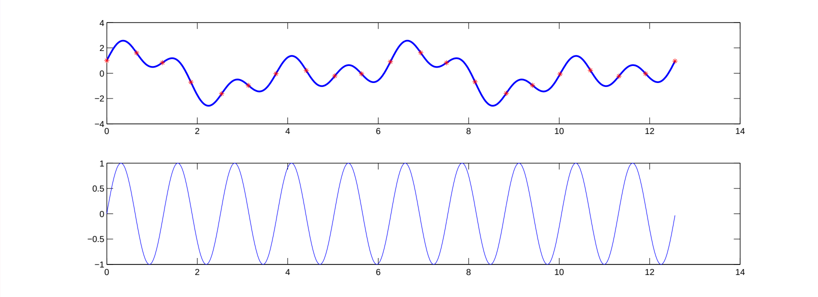

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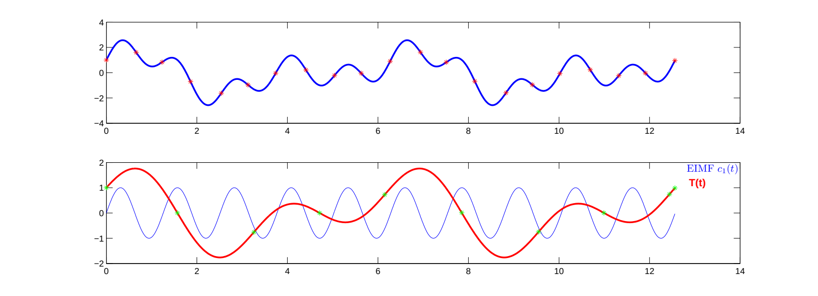


Figure: We extract the first EIMF in the same time we ensure that $T(t)$ is slower than $c(t)$

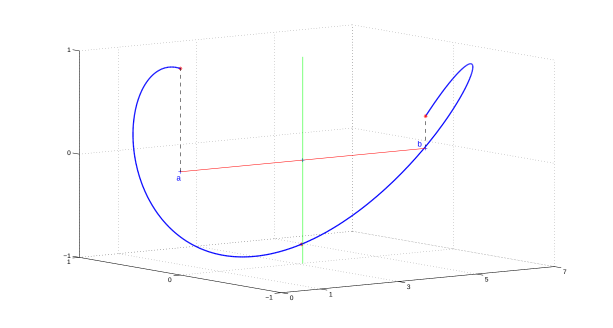
Outlines

Elementary Intrinsic Mode Functions (EIMF) the 2D case

A bivariate EIMF is a bivariate function having one of the following shapes:

- Rotating EIMFs: they are non-planar curves that are turning around the time axis (they have a spiral shape) in addition they are locally axially symmetric.
- Oscillating EIMFs: it is a planar curve having many changes of the convexity and have a "local symmetry" as well. In their containing plan, they can be seen as univariate EIMFs.
- Tendencies: they are planar or non planar curves which have no inflexion.

A simple illustration: the 2D case



For $\overrightarrow{f}(t) = (x(t), y(t))$, the idea is to search for $\overrightarrow{T}(t) = (T_1(t), T_2(t))$ such that $\overrightarrow{f}(t) - \overrightarrow{T}(t) = \overrightarrow{c}(t)$ be a bivariate EIMF on $[a, b]$ and it verifies the axial symmetry with respect to the line passing by $(\frac{a+b}{2}, 0, 0)$ and parallels to a reference $\overrightarrow{k}$.

The existence, uniqueness and convergence

As in univariate signals we show that regular function $\overrightarrow{f}(t)$ may be locally decomposed on $[a, b]$ as: $\overrightarrow{f}(t) = \overrightarrow{c}(t) + \overrightarrow{T}(t)$ where:

$$c_1(t) = \frac{x(t) - x(a+b-t)}{2} - \varphi(2t - (a+b))$$
$$c_2(t) = \frac{y(t) + y(a+b-t)}{2} - \psi(2t - (a+b))$$

φ is odd and ψ is even and are linked to $\overrightarrow{T}(t)$.

We also show that $(\overrightarrow{c}(t), \overrightarrow{T}(t))$ is the unique such that $\overrightarrow{c}(t)$ is an EIMF and $\overrightarrow{T}(t)$ do not change the convexity on $[a, b]$

Inline extraction in a general bivariate signal

Theorem

Let $\overrightarrow{f}(t) = (x(t), y(t))$ be a regular function having $m = 2\ell - 1$ inflexion points at $a_1, \ldots, a_m$. There exist, a set of odd functions $(\varphi_1, \ldots, \varphi_\ell)$ and a set of even functions $(\psi_1, \ldots, \psi_\ell)$ such that $\overrightarrow{f}(t) = \overrightarrow{c}(t) + \overrightarrow{T}(t)$ where

$$c_1(t) = \sum_{i=1}^{\ell} \frac{x(t) - x(a_{2i-1} + a_{2i+1} - t)}{2} - \varphi_i(2t - (a_{2i-1} + a_{2i+1}))$$

$$c_2(t) = \sum_{i=1}^{\ell} \frac{y(t) + y(a_{2i-1} + a_{2i+1} - t)}{2} - \psi_i(2t - (a_{2i-1} + a_{2i+1}))$$

and $\overrightarrow{T}$ is a function having at most ℓ inflexion points.

The main algorithms: characteristic points

Algorithm 1: Detection of characteristic points

Input: Bivariate signal $\overrightarrow{f}(t_i) = (x(t_i), y(t_i))$ observed at $t_i, i = 1, \ldots, n$

- Calculate first and second derivatives of $x(t)$, $y(t)$

foreach $i = 1 \ldots n$ **do**

- Calculate the vector product

$$\overrightarrow{U} = \begin{pmatrix} 1 \\ x'(t_i) \\ y'(t_i) \end{pmatrix} \wedge \begin{pmatrix} 0 \\ x''(t_i) \\ y''(t_i) \end{pmatrix}$$

- Calculate the dot product $Cu(t) = \overrightarrow{U} \cdot \overrightarrow{k}$

end

Find the zeros crossing of $Cu(t)$, $\theta_1, \ldots, \theta_{2m-1}$

Take characteristic points as $a_i = \theta_{2i-1}$ for $i = 1, \ldots, m$

Output: Characteristic points $a_1, \ldots a_m$

Main algorithms: The inline extraction

$$\varphi_i(t) = \alpha_{1,i} t + \alpha_{2,i} t^3 + \alpha_{3,i} t^5,$$
$$\psi_i(t) = \beta_{1,i} + \beta_{2,i} t^2 + \beta_{3,i} t^4.$$

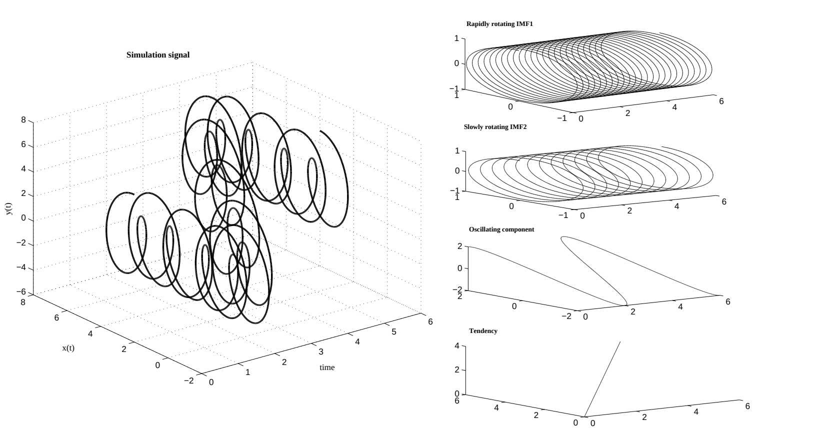
The coefficients in α and β are obtained by a piecewise spline like procedure as in the univariate case.

Algorithm 2: Extraction of rapidly rotating and rapidly oscillating IMFs

Input: Bivariate signal $\overrightarrow{f}(t_i) = (x(t_i), y(t_i))$ observed at $t_i, i = 1 : n$
Find characteristic points $a_1, \dots a_m$ using algorithm1
foreach $i = 1 \dots m - 1$ **do**
- Find coefficients of φ_i and ψ_i on $[a_i, a_{i+1}]$
- Calculate c_1 and c_2 as given in the decomposition theorem.

end
Output: Elementary IMFs $\overrightarrow{c}(t) = (c_1(t), c_2(t))$

A simulated example



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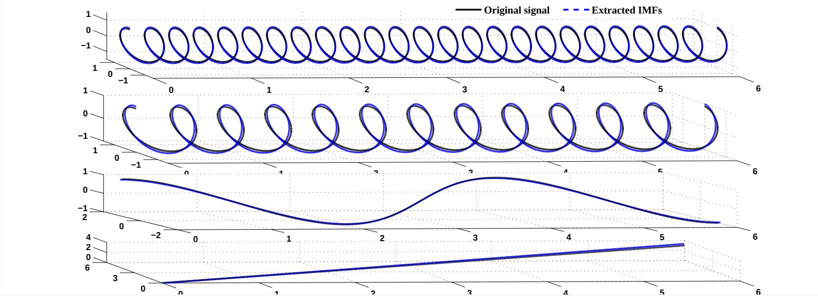

Perspectives and further issues

- Study the sensitivity to sampling and perturbations
- Generalizing the technique to higher order vectorial and multivariate signals
- Use the discrete convexity discrete geometry concepts
- Study of the hidden scales problems