A generalization of the Poincaré-Miranda theorem with an application to the controllability of nonlinear repetitive processes[1]

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Theorem (Bolzano)

If a function $f : [-L, L] \to \mathbb{R}$ is continuous and such that

$$f(-L) \leqslant 0 \qquad \textit{and} \qquad f(L) \geqslant 0$$

then there exist at least one point $x_* \in [-L, L]$ such that $f(x_*) = 0$.

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This theorem was first stated by Bernard Bolzano (1781 – 1848) in 1817

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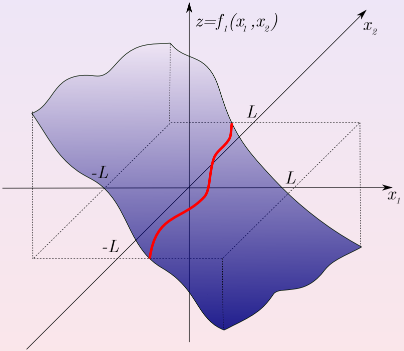
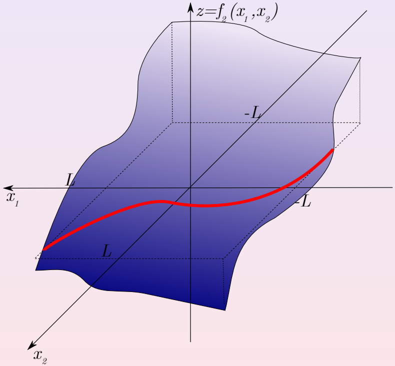
Theorem (Poincaré-Miranda I)

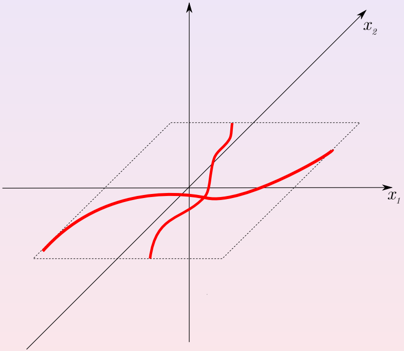
Let $Q := [-L, L] \times \dots \times [-L, L]$ where $L > 0$ is a fixed number, be an interval in $\mathbb{R}^n$. If a continuous function $f = (f_1, \dots, f_n) : Q \to \mathbb{R}^n$ is such that for any $i = 1, \dots, n$

$$f_i(x) \geqslant 0 \quad \text{for each } x \in Q_i^- := \{x = (x_1, \dots, x_n) \in Q;\ x_i = -L\},$$
$$f_i(x) \leqslant 0 \quad \text{for each } x \in Q_i^+ := \{x = (x_1, \dots, x_n) \in Q;\ x_i = L\}$$

then there exist at least one point $x_* \in Q$ such that $f(x_*) = 0$.


z=f1(x1,x2)
x2
L
-L
L
x1
-L


$z = f_2(x_1, x_2)$
-L
L
x_1
-L
L
x_2


x_2
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- the classical Tikhonov topology τ.

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Convergence of a sequence

$$(z^{(n)})_{n \in \mathbb{N}}$$

to a point $z^{(0)}$ in the space $\mathcal{R}$ means that

$$z_i^{(n)} \longrightarrow z_i^{(0)}$$

for any $i \in \mathbb{N}$.

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Theorem (Poincaré-Miranda II)

If a continuous function $f = (f_1, f_2, \ldots) : \mathcal{Q} \to \mathcal{R}$ is such that for any $i \in \mathbb{N}$

$$\begin{aligned} f_i(x) \geqslant 0 \quad & \text{for each } x \in \mathcal{Q}_i^- := \{x = (x_1, x_2, \ldots) \in \mathcal{Q};\ x_i = -L\}, \\ f_i(x) \leqslant 0 \quad & \text{for each } x \in \mathcal{Q}_i^+ := \{x = (x_1, x_2, \ldots) \in \mathcal{Q};\ x_i = L\}, \end{aligned}$$

then there exists at least one point $x_* \in \mathcal{Q}$ such that $f(x_*) = 0$.

Consider the following nonlinear process

$$\begin{cases} \frac{d}{dt} z_{k+1}(t) = f^1(t, z_{k+1}(t), w_k(t), u_{k+1}(t)) \\ w_{k+1}(t) = f^2(t, z_{k+1}(t), w_k(t), u_{k+1}(t)) \end{cases} \tag{1}$$

for $k \in \mathbb{N} \cup \{0\}$, $t \in [0, \alpha]$ a.e. ($\alpha > 0$ is a fixed number),

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for $k \in \mathbb{N} \cup \{0\}$, $t \in [0, \alpha]$ a.e. ($\alpha > 0$ is a fixed number),
with the initial conditions

$$\begin{cases} z_k(0) = d_k, \ \ k \in \mathbb{N} \\ w_0(t) = h(t), \ \ t \in [0, \alpha] \text{ a.e.} \end{cases} \qquad (2)$$

where
$z_k(t) \in \mathbb{R}$, $w_k(t) \in \mathbb{R}$, $u_k(t) \in \mathbb{R}$,

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$u_k(\cdot)$ – a control on a pass k,
$z_k(\cdot)$ – a trajectory on a pass k
$w_k(\cdot)$ – an output on a pass k.

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A_f. f^i is measurable in $t \in [0, \alpha]$, continuous in $u \in \Omega$ and lipschitzian in $(z, w) \in \mathbb{R} \times \mathbb{R}$, i.e. there exists a constant $K_i > 0$ such that

$$\left| f^i(t, z, w, u) - f^i(t, \overline{z}, \overline{w}, u) \right| \leqslant K_i(|z - \overline{z}| + |w - \overline{w}|)$$

for $t \in [0, \alpha]$ a.e., z, $\overline{z} \in \mathbb{R}$, w, $\overline{w} \in \mathbb{R}$, $u \in \Omega$,

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B_{f^1}. for any measurable function $u : [0,\alpha] \to \Omega$ the function $f^1(\cdot,0,0,u(\cdot))$ is integrable on $[0,\alpha]$.

$AC([0,\alpha],\mathcal{R})$ – the space of functions

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The spaces $AC([0,\alpha],\mathcal{R})$, $L^1([0,\alpha], \Omega \times \Omega \times ...)$ will be considered with Tikhonov topologies.

Theorem

If the functions f^1, f^2 satisfy conditions A_f, B_{f^1}, then for any initial points $d_i \in \mathbb{R}$, $i \in \mathbb{N}$, measurable function $h : [0, \alpha] \longrightarrow \mathbb{R}$, and for any control $(u_1(\cdot), u_2(\cdot), ...)$ with measurable coordinate functions $u_i : [0, \alpha] \longrightarrow \Omega$, $i \in \mathbb{N}$, repetitive process (1)-(2) has a unique solution $(z_1(\cdot), z_2(\cdot), ...) \in AC([0, \alpha], \mathcal{R})$.

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Moreover, if a sequence of controls $(u^{(n)}(\cdot)) = (u_1^{(n)}(\cdot), u_2^{(n)}(\cdot), \ldots)$ converges in $L^1([0, \alpha], \Omega \times \Omega \times \ldots)$ to some $u^0(\cdot)$, then $z_i^{(n)}(\cdot) \longrightarrow z_i^0(\cdot)$ in $AC([0, \alpha], \mathbb{R})$ for $i \in \mathbb{N}$, where $(z_1^{(n)}(\cdot), z_2^{(n)}(\cdot), \ldots)$ is the trajectory corresponding to control $(u_1^{(n)}(\cdot), u_2^{(n)}(\cdot), \ldots)$, $n \in \mathbb{N} \cup \{0\}$.

Since the convergence with respect to norm in $AC([0,\alpha],\mathbb{R})$ implies the pointwise one, therefore

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Theorem

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$$\Phi : L^1([0,\alpha], \Omega \times \Omega \times \ldots) \ni u(\cdot) = (u_1(\cdot), u_2(\cdot), \ldots) \longmapsto z^u(\alpha) = (z_1^u(\alpha), z_2^u(\alpha), \ldots) \in \mathcal{R}$$

is continuous (here $z^u(\cdot)$ is the solution of process (1)-(2), corresponding to control $u(\cdot)$).

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Consider the set

$$U = \{u(\cdot) = \sum_{i=1}^{\infty} \beta_i u^{(i)}(\cdot);\ \beta = (\beta_1, \beta_2, \dots) \in \mathcal{R}\} = \\ \{u(\cdot) = (\beta_1 u_1(\cdot), \beta_2 u_2(\cdot), \dots); \beta = (\beta_1, \beta_2, \dots) \in \mathcal{R}\}..$$

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Let $\kappa : \mathcal{R} \to L^1([0,\alpha], \mathcal{R})$ be a mapping given by

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$$\kappa(\beta_1, \beta_2, \ldots) = (\beta_1 u_1(\cdot), \beta_2 u_2(\cdot), \ldots).$$

Of course, it is continuous.

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Consequently, the mapping $\varphi = \Phi \circ (\kappa\,|_P) = (\varphi_1, \varphi_2, \ldots) : P \to \mathcal{R}$

$$\varphi_i(\beta_1, \ldots, \beta_{i-1}, \beta_i, \beta_{i+1}, \ldots) = z_i^{(\beta_1 u_1(\cdot), \beta_2 u_2(\cdot), \ldots)}(\alpha)$$

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is well-defined and continuous.

Sufficient condition for the controllability

Let us assume that the functions f^1, f^2 satisfy conditions A_f, B_{f1} and initial points $d_i \in \mathbb{R}$, $i \in \mathbb{N}$, and measurable function $h : [0, \alpha] \to \mathbb{R}$ are given.

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If $m_i \leqslant M_i$ for $i \in \mathbb{N}$, where

$$\begin{aligned} M_i &:= \min\{\varphi_i(\beta_1, ..., \beta_{i-1}, a_i, \beta_{i+1}, ...); (\beta_1, ..., \beta_{i-1}, a_i, \beta_{i+1}, ...) \in P\}, \\ m_i &:= \max\{\varphi_i(\beta_1, ..., \beta_{i-1}, b_i, \beta_{i+1}, ...); (\beta_1, ..., \beta_{i-1}, b_i, \beta_{i+1}, ...) \in P\}, \end{aligned}$$

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then for any point $z \in [m_1, M_1] \times [m_2, M_2] \times ... \subset \mathcal{R}$ there exist $\beta_i \in [a_i, b_i]$, $i \in \mathbb{N}$, such that

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then for any point $z \in [m_1, M_1] \times [m_2, M_2] \times ... \subset \mathcal{R}$ there exist $\beta_i \in [a_i, b_i]$, $i \in \mathbb{N}$, such that

$$z^{(\beta_1 u_1(\cdot), \beta_2 u_2(\cdot), ...)}(\alpha) = z.$$

$$\begin{cases} \frac{d}{dt} z_{i+1}(t) = z_{i+1}(t) - w_i(t) \cdot u_{i+1}(t) \\ w_{i+1}(t) = z_{i+1}(t) \end{cases}$$

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Let

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$$P = [-1, 1] \times [-1, 1] \times \ldots \subset \mathcal{R}$$

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Let

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and fix the functions

$$u_i : [0, 1] \ni t \longmapsto 1 \in \Omega$$

for $i \in \mathbb{N}$.

In our case

$$(\beta_1 u_1(\cdot), \beta_2 u_2(\cdot), \ldots) \equiv (\beta_1, \beta_2, \ldots)$$

and

$$\varphi_i(\beta_1, \ldots, \beta_{i-1}, \beta_i, \beta_{i+1}, \ldots) = z_i^{(\beta_1 u_1(\cdot), \beta_2 u_2(\cdot), \ldots)}(1) = z_i^{(\beta_1, \beta_2, \ldots)}(1) = z_i^{(\beta_1, \ldots, \beta_i)}(1)$$

for $i \in \mathbb{N}$.

$$
\begin{aligned}
M_i &= \min\{\varphi_i(\beta_1, ..., \beta_{i-1}, -1, \beta_{i+1}, ...); (\beta_1, ..., \beta_{i-1}, -1, \beta_{i+1}, ...) \in P\} \\
&= \min\{z_i^{\beta_1, \dots, \beta_{i-1}, -1}(1); (\beta_1, ..., \beta_{i-1}, -1) \in [-1, 1] \times ... \times [-1, 1]\} \\
&\geqslant e(4 - \textstyle\sum_{k=0}^{i-1} \frac{1}{k!}) > e(4 - e) > e(-2 + e) > e(-2 + \textstyle\sum_{k=0}^{i-1} \frac{1}{k!}) \\
&\geqslant \max\{z_i^{\beta_1, \dots, \beta_{i-1}, 1}(1); (\beta_1, ..., \beta_{i-1}, 1) \in [-1, 1] \times ... \times [-1, 1]\} \\
&= \max\{\varphi_i(\beta_1, ..., \beta_{i-1}, 1, \beta_{i+1}, ...); (\beta_1, ..., \beta_{i-1}, 1, \beta_{i+1}, ...) \in P\} = m_i
\end{aligned}
$$

for $i \geqslant 2$.

$$
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&\geqslant \max\{z_i^{\beta_1,\ldots,\beta_{i-1},1}(1); (\beta_1, ..., \beta_{i-1}, 1) \in [-1,1] \times ... \times [-1,1]\} \\
&= \max\{\varphi_i(\beta_1, ..., \beta_{i-1}, 1, \beta_{i+1}, ...); (\beta_1, ..., \beta_{i-1}, 1, \beta_{i+1}, ...) \in P\} = m_i
\end{aligned}
$$

for $i \geqslant 2$.

$$M_1 = m_1.$$

Thus, the sufficient condition implies that for any point

$$z \in [m_1, M_1] \times [m_2, M_2] \times \ldots \subset \mathcal{R}$$

there exist $\beta_i \in [-1, 1]$, $i \in \mathbb{N}$,

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In particular, any point

$$z = (z_1, z_2, z_3, \dots) \in [e, e] \times [e(-2+e), e(4-e)] \times [e(-2+e), e(4-e)] \times \dots$$

has the above property.

Thank you for your attention