

On the Kalman filter for 2-D Fornasini-Marchesini models

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Outline

- Problem Formulation
- A Geometric Approach
- Structure of the 2-D Kalman filter
- Gain matrices of the optimal filter
- A simplified filter
- An example
- Conclusions and future works

Problem Formulation: The FM Model

Consider the Fornasini-Marchesini Model

$$x_{i+1,j+1} = A_{i+1,j}^{(1)} x_{i+1,j} + A_{i,j+1}^{(2)} x_{i,j+1} + B_{i+1,j}^{(1)} w_{i+1,j} + B_{i,j+1}^{(2)} w_{i,j+1}$$

$$y_{i,j} = C_{i,j} x_{i,j} + v_{i,j}$$

where for all i, j

- $x_{i,j} \in \mathbb{C}^n$ is the *local state*;
- $y_{i,j} \in \mathbb{C}^p$ is the measurement vector;
- $w_{i,j} \in \mathbb{C}^m$ and $v_{i,j} \in \mathbb{C}^p$ are the plant and measurement noises.

Problem Formulation: The FM Model

Consider the Fornasini-Marchesini Model

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$$y_{i,j} = C_{i,j} x_{i,j} + v_{i,j}$$

Boundary conditions:



Problem Formulation: Assumptions

The boundary conditions $x_{i,0}$, $x_{0,j}$ and the noises $w_{i,j}$, $v_{i,j}$ are assumed to be zero mean random variables with

$$\left\langle \begin{bmatrix} w_{i,j} \\ v_{i,j} \\ x_{i,0} \\ x_{0,j} \end{bmatrix}, \begin{bmatrix} w_{k,l} \\ v_{k,l} \\ x_{k,0} \\ x_{0,l} \end{bmatrix} \right\rangle = \begin{bmatrix} \begin{bmatrix} Q_{i,j} & S_{i,j} \\ S_{i,j}^* & R_{i,j} \end{bmatrix} \delta_{i,k} \delta_{j,l} & 0 \\ 0 & \begin{bmatrix} \Pi_{i,0} \delta_{i,k} & 0 \\ 0 & \Pi_{0,j} \delta_{j,l} \end{bmatrix} \end{bmatrix}$$

where

$$\delta_{i,k} = \begin{cases} 1 & \text{if } i = k \\ 0 & \text{if } i \neq k \end{cases}$$

It is assumed that $R_{i,j} = R_{i,j}^* > 0$ for all i and j .

Problem Formulation

Given the measurements

$$\{y_{k,l} \in \mathbb{C}^p \mid 0 \leq (k,l) < (i,j)\}$$

find an estimation $\hat{x}_{i,j}$ of $x_{i,j}$ with $\hat{x}_{k,0} = 0$ and $\hat{x}_{0,l} = 0$, such that

$$E\{\tilde{x}_{i,j}^* \tilde{x}_{i,j}\}$$

is minimised, where $\tilde{x}_{i,j} := x_{i,j} - \hat{x}_{i,j}$ represents the estimation error.



A Least-Mean-Squares Approach to Kalman Filtering

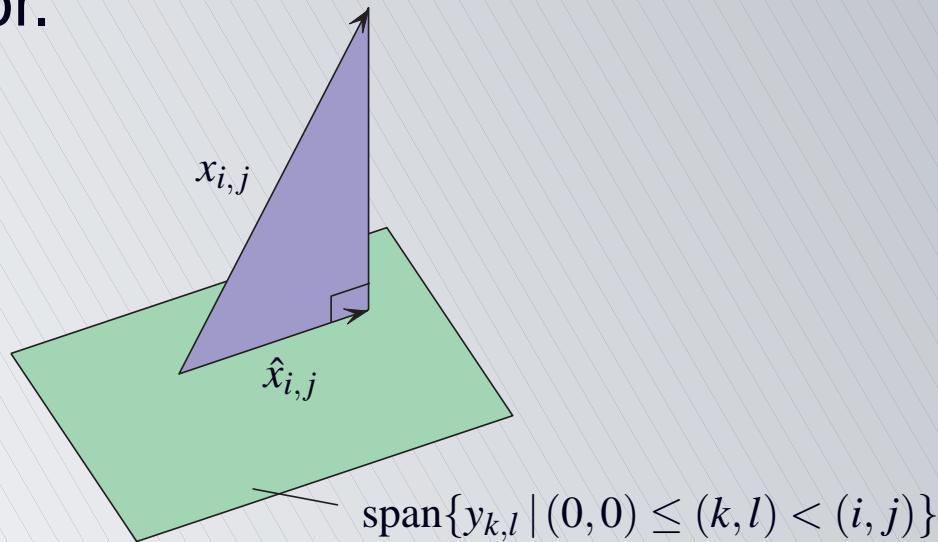
The local state of the least-mean-square estimator which minimises $\langle \tilde{x}_{i,j}, \tilde{x}_{i,j} \rangle$, is

$$\hat{x}_{i,j} = \sum_{(0,0) \leq (k,l) < (i,j)} \langle x_{i,j}, e_{k,l} \rangle \langle e_{k,l}, e_{k,l} \rangle^{-1} e_{k,l}$$

where

$$e_{k,l} = y_{k,l} - C_{k,l} \hat{x}_{k,l}$$

is the estimation error.



A Least-Mean-Squares Approach to Kalman Filtering

The 2-D filter ruled by

$$\begin{aligned}\hat{x}_{i+1,j+1} &= A_{i+1,j}^{(1)} \hat{x}_{i+1,j} + A_{i,j+1}^{(2)} \hat{x}_{i,j+1} \\ &\quad + \sum_{l=0}^j K_{i+1,j+1;l}^{(1)} e_{i+1,l} + \sum_{k=0}^i K_{i+1,j+1;k}^{(2)} e_{k,j+1} \\ e_{i,j} &= y_{i,j} - C_{i,j} \hat{x}_{i,j}\end{aligned}$$

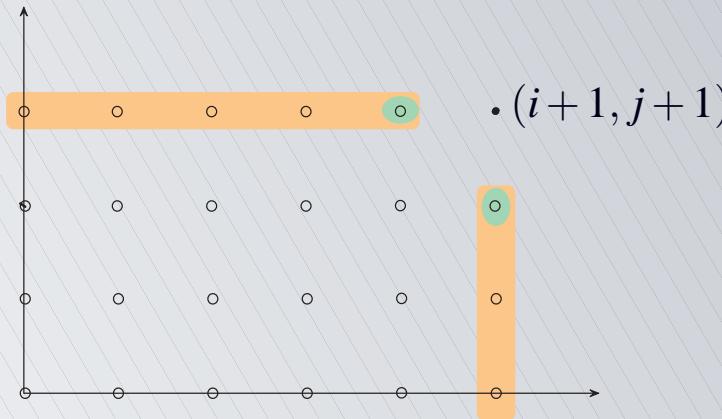
with boundary conditions $\hat{x}_{l,0} = 0$ and $\hat{x}_{0,k} = 0$ minimises

$$P_{i,j} := \langle \tilde{x}_{i,j}, \tilde{x}_{i,j} \rangle = \langle x_{i,j} - \hat{x}_{i,j}, x_{i,j} - \hat{x}_{i,j} \rangle$$

A Least-Mean-Squares Approach to Kalman Filtering

The 2-D filter ruled by

$$\begin{aligned}\hat{x}_{i+1,j+1} &= A_{i+1,j}^{(1)} \hat{x}_{i+1,j} + A_{i,j+1}^{(2)} \hat{x}_{i,j+1} \\ &\quad + \sum_{l=0}^j K_{i+1,j+1;l}^{(1)} e_{i+1,l} + \sum_{k=0}^i K_{i+1,j+1;k}^{(2)} e_{k,j+1} \\ e_{i,j} &= y_{i,j} - C_{i,j} \hat{x}_{i,j}\end{aligned}$$



Gain Matrices of the 2-D Kalman Filter

By defining $T_{i,j} := C_{i,j} P_{i,j} C_{i,j}^* + R_{i,j}$, the gains of the filter

$$\begin{aligned}\hat{x}_{i+1,j+1} &= A_{i+1,j}^{(1)} \hat{x}_{i+1,j} + A_{i,j+1}^{(2)} \hat{x}_{i,j+1} \\ &\quad + \sum_{l=0}^j K_{i+1,j+1;l}^{(1)} e_{i+1,l} + \sum_{k=0}^i K_{i+1,j+1;k}^{(2)} e_{k,j+1} \\ e_{i,j} &= y_{i,j} - C_{i,j} \hat{x}_{i,j}\end{aligned}$$

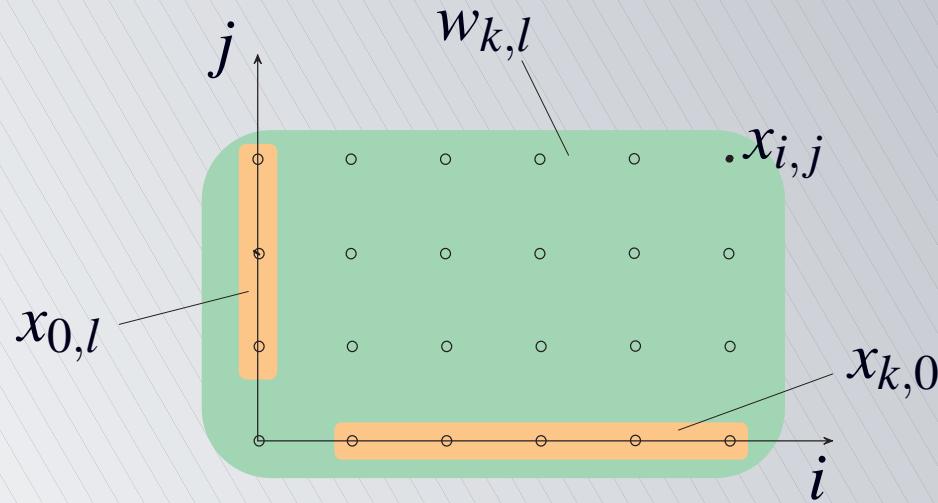
can be expressed as

$$\begin{aligned}K_{i+1,j+1;l}^{(1)} &= \begin{cases} \left(A_{i,j+1}^{(2)} \langle x_{i,j+1}, \tilde{x}_{i+1,l} \rangle C_{i+1,l}^* + A_{i+1,j}^{(1)} P_{i+1,j} C_{i+1,j}^* \right) T_{i+1,l}^{-1} & l = j \\ A_{i,j+1}^{(2)} \langle x_{i,j+1}, \tilde{x}_{i+1,l} \rangle C_{i+1,l}^* T_{i+1,l}^{-1} & l = 0, \dots, j-1 \end{cases} \\ K_{i+1,j+1;k}^{(2)} &= \begin{cases} \left(A_{i+1,j}^{(1)} \langle x_{i+1,j}, \tilde{x}_{k,j+1} \rangle C_{k,j+1}^* + A_{i,j+1}^{(2)} P_{i,j+1} C_{i,j+1}^* \right) T_{k,j+1}^{-1} & k = i; \\ A_{i+1,j}^{(1)} \langle x_{i+1,j}, \tilde{x}_{k,j+1} \rangle C_{k,j+1}^* T_{k,j+1}^{-1} & k = 0, \dots, i-1. \end{cases}\end{aligned}$$

Gain Matrices: Result 1

In the FM model, the state $x_{i,j}$ can be written as

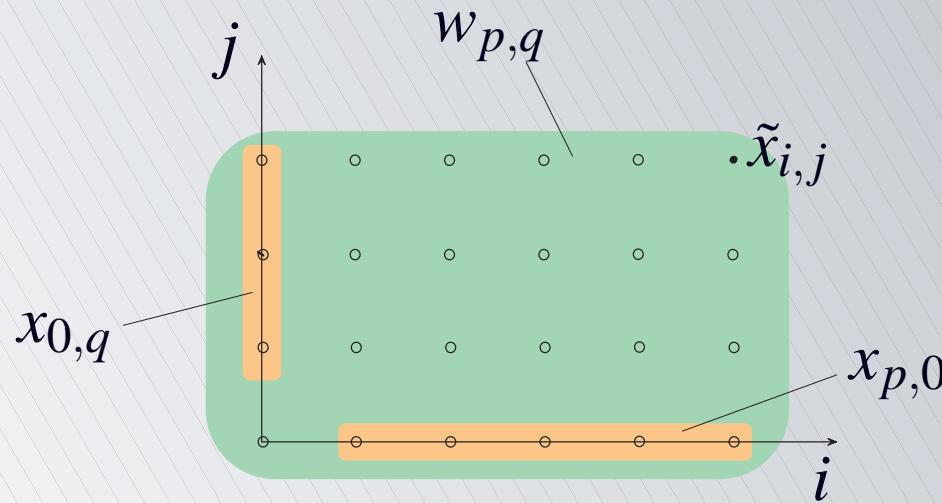
$$x_{i,j} = \sum_{k=1}^i \Phi_{i,j;k,0} x_{k,0} + \sum_{l=1}^j \Phi_{i,j;0,l} x_{0,l} + \sum_{k=0}^i \sum_{l=0}^j \Psi_{i,j;k,l} w_{k,l}$$



Gain Matrices: Result 2

The local state estimation error can be written as

$$\begin{aligned}\tilde{x}_{i,j} = & \sum_{q=1}^j \Theta_{i,j;0,q} x_{0,q} + \sum_{p=1}^i \Theta_{i,j;p,0} x_{p,0} \\ & + \sum_{p=0}^i \sum_{q=0}^j \Xi_{i,j;p,q} w_{p,q} + \sum_{p=0}^i \sum_{q=0}^j \Delta_{i,j;p,q} v_{p,q}\end{aligned}$$



Gain Matrices: Result 3

We can write the products

$$K_{i+1,j+1;l}^{(1)} = \begin{cases} \left(A_{i,j+1}^{(2)} \langle x_{i,j+1}, \tilde{x}_{i+1,l} \rangle C_{i+1,l}^* + A_{i+1,j}^{(1)} P_{i+1,j} C_{i+1,j}^* \right) T_{i+1,l}^{-1} & l = j \\ A_{i,j+1}^{(2)} \langle x_{i,j+1}, \tilde{x}_{i+1,l} \rangle C_{i+1,l}^* T_{i+1,l}^{-1} & l = 0, \dots, j-1 \end{cases}$$

$$K_{i+1,j+1;k}^{(2)} = \begin{cases} \left(A_{i+1,j}^{(1)} \langle x_{i+1,j}, \tilde{x}_{k,j+1} \rangle C_{k,j+1}^* + A_{i,j+1}^{(2)} P_{i,j+1} C_{i,j+1}^* \right) T_{k,j+1}^{-1} & k = i; \\ A_{i+1,j}^{(1)} \langle x_{i+1,j}, \tilde{x}_{k,j+1} \rangle C_{k,j+1}^* T_{k,j+1}^{-1} & k = 0, \dots, i-1. \end{cases}$$

as follows:

$$\begin{aligned} \langle x_{i,j+1}, \tilde{x}_{i+1,l} \rangle &= \sum_{p=1}^i \Phi_{i,j+1;p,0} \Pi_{p,0} \Theta_{i+1,l;p,0}^* + \sum_{q=1}^l \Phi_{i,j+1;0,q} \Pi_{0,q} \Theta_{i+1,l;0,q}^* \\ &\quad + \sum_{p=0}^i \sum_{q=0}^l \Psi_{i,j+1;p,q} Q_{p,q} \Xi_{i+1,l;p,q}^* + \sum_{p=0}^i \sum_{q=0}^l \Psi_{i,j+1;p,q} S_{p,q} \Delta_{i+1,l;p,q}^*. \end{aligned}$$

Gain Matrices: Result 3

Likewise, we can write the products

$$K_{i+1,j+1;l}^{(1)} = \begin{cases} \left(A_{i,j+1}^{(2)} \langle x_{i,j+1}, \tilde{x}_{i+1,l} \rangle C_{i+1,l}^* + A_{i+1,j}^{(1)} P_{i+1,j} C_{i+1,j}^* \right) T_{i+1,l}^{-1} & l = j \\ A_{i,j+1}^{(2)} \langle x_{i,j+1}, \tilde{x}_{i+1,l} \rangle C_{i+1,l}^* T_{i+1,l}^{-1} & l = 0, \dots, j-1 \end{cases}$$
$$K_{i+1,j+1;k}^{(2)} = \begin{cases} \left(A_{i+1,j}^{(1)} \langle x_{i+1,j}, \tilde{x}_{k,j+1} \rangle C_{k,j+1}^* + A_{i,j+1}^{(2)} P_{i,j+1} C_{i,j+1}^* \right) T_{k,j+1}^{-1} & k = i; \\ A_{i+1,j}^{(1)} \langle x_{i+1,j}, \tilde{x}_{k,j+1} \rangle C_{k,j+1}^* T_{k,j+1}^{-1} & k = 0, \dots, i-1. \end{cases}$$

as follows:

$$\begin{aligned} \langle x_{i+1,j}, \tilde{x}_{k,j+1} \rangle &= \sum_{p=1}^k \Phi_{i+1,j;p,0} \Pi_{p,0} \Theta_{k,j+1;p,0}^* + \sum_{q=1}^j \Phi_{i+1,j;0,q} \Pi_{0,q} \Theta_{k,j+1;0,q}^* \\ &\quad + \sum_{p=0}^k \sum_{q=0}^j \Psi_{i+1,j;p,q} Q_{p,q} \Xi_{k,j+1;p,q}^* + \sum_{p=0}^k \sum_{q=0}^j \Psi_{i+1,j;p,q} S_{p,q} \Delta_{k,j+1;p,q}^*. \end{aligned}$$

Gain Matrices: Result 4

Matrix $P_{i,j}$ can be computed as

$$\begin{aligned} P_{i,j} = & \left(\sum_{q=1}^{j-1} \Theta_{i,j;0,q} \Pi_{0,q} \Phi_{i,j-1;0,q}^* + \sum_{p=1}^i \Theta_{i,j;p,0} \Pi_{p,0} \Phi_{i,j-1;p,0}^* \right. \\ & + \sum_{p=0}^i \sum_{q=0}^{j-1} (\Xi_{i,j;p,q} Q_{p,q} + \Delta_{i,j;p,q} S_{p,q}) \Psi_{i,j-1;p,q}^* \Big) (A_{i,j-1}^{(1)})^* \\ & + \left(\sum_{q=1}^j \Theta_{i,j;0,q} \Pi_{0,q} \Phi_{i-1,j;0,q}^* + \sum_{p=1}^{i-1} \Theta_{i,j;p,0} \Pi_{p,0} \Phi_{i-1,j;p,0}^* \right. \\ & + \sum_{p=0}^{i-1} \sum_{q=0}^j (\Xi_{i,j;p,q} Q_{p,q} + \Delta_{i,j;p,q} S_{p,q}) \Psi_{i-1,j;p,q}^* \Big) (A_{i-1,j}^{(2)})^* \\ & + B_{i,j-1}^{(1)} Q_{i,j-1} (B_{i,j-1}^{(1)})^* + B_{i-1,j}^{(2)} Q_{i-1,j} (B_{i-1,j}^{(2)})^*. \end{aligned}$$

A Simplified Filter

Consider the filter

$$\begin{aligned}\hat{x}_{i+1,j+1} &= A_{i+1,j}^{(1)} \hat{x}_{i+1,j} + A_{i,j+1}^{(2)} \hat{x}_{i,j+1} + K_{i+1,j+1;j}^{(1)} e_{i+1,j} + K_{i+1,j+1;i}^{(2)} e_{i,j+1} \\ e_{i,j} &= y_{i,j} - C_{i,j} \hat{x}_{i,j}\end{aligned}$$

with boundary conditions $\hat{x}_{k,0} = 0$ and $\hat{x}_{0,l} = 0$, where

$$\begin{aligned}K_{i+1,j+1;j}^{(1)} &= \left(A_{i,j+1}^{(2)} \langle x_{i,j+1}, \tilde{x}_{i+1,j} \rangle C_{i+1,j}^* + A_{i+1,j}^{(1)} P_{i+1,j} C_{i+1,j}^* \right) T_{i+1,j}^{-1} \\ K_{i+1,j+1;i}^{(2)} &= \left(A_{i+1,j}^{(1)} \langle x_{i+1,j}, \tilde{x}_{i,j+1} \rangle C_{i,j+1}^* + A_{i,j+1}^{(2)} P_{i,j+1} C_{i,j+1}^* \right) T_{i,j+1}^{-1}\end{aligned}$$

The difference lies in the structure of the filter, and implicitly in the computation of the matrices $\Theta_{i,j;p,q}$, $\Xi_{i,j;p,q}$ and $\Delta_{i,j;p,q}$.

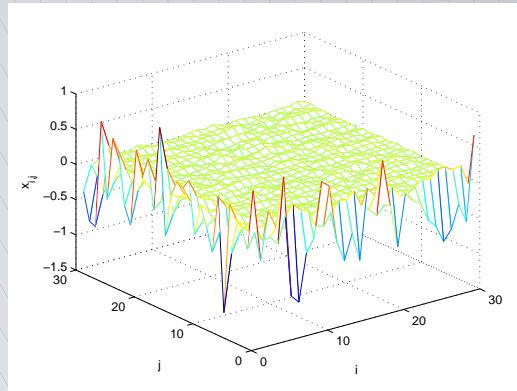
An Example

Consider a FM model where

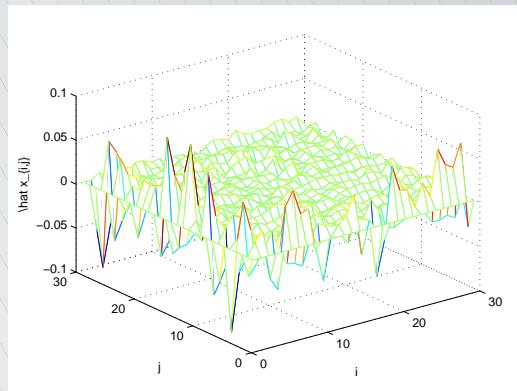
- $A_{i,j}^{(1)} = 0.3 + 0.1 \sin(0.01 i^2 + 0.01 j^2),$
 $A_{i,j}^{(2)} = -0.25,$
 $B_{i,j}^{(1)} = 2, B_{i,j}^{(2)} = -0.1 \sin(0.01 j),$
 $C_{i,j} = 0.2;$
- the boundary conditions $x_{i,0}$ and $x_{0,j}$, and the noises $w_{i,j}$ and $v_{i,j}$ are assumed to be zero mean random variables with covariance matrices
 $Q_{i,j} = 0.01, S_{i,j} = 0, R_{i,j} = 0.06,$
 $\Pi_{i,0} = 0.5$ and $\Pi_{0,j} = 0.5.$

An Example

Local state $x_{i,j}$:



Estimated local state $\hat{x}_{i,j}$:

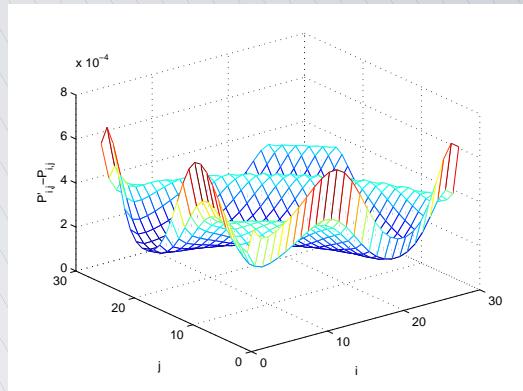


Comparison with the simplified filter

The simplified filter is compared against the 2-D Kalman filter.

- $P_{i,j}$: variance of the error of the 2-D Kalman filter;
- $P'_{i,j}$: variance of the error of the simplified filter.

The figure below shows the difference $P'_{i,j} - P_{i,j}$. Obviously $P'_{i,j}$ is greater than $P_{i,j}$.



Concluding Remarks

- The filter that generates the best estimate of the local state (in the sense of the Least-Means-Squares) is not governed by a Fornasini-Marchesini model;
- Optimality is guaranteed in a necessary and sufficient sense;
- The size of the recursion increases as the index evolves away from the boundary;
- Persistence is necessary to obtain computationally tractable expressions for the gains of the filter.

Future Works

Future investigations include:

- friendlier expressions for the gain matrices of the optimal filter;
- a shift-invariant version of the optimal filter;
- a comparison between the efficiency of the simplified filter against the optimal one;
- a comparison between the optimal filter and the suboptimal ones proposed in the literature.

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