On some nonlinear second order control systems

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We consider the second order equation

$$\ddot{x}(t) = \frac{1}{m(t)} G(t, x(t)) w(t) + \frac{1}{m(t)} C(t) u(t) \tag{1}$$

with the boundary conditions

$$x(t_0) = \bar{x}_0, x(t_1) = \bar{x}_1 \tag{2}$$

- $G : [t_0, t_1] \times \mathbb{R}^n \to \mathbb{R}^{n \times K}$, $C : [t_0, t_1] \to \mathbb{R}^{n \times N}$ are given matrix-valued functions
- w and u are controls such that for $t \in [t_0, t_1]$

$$w(t) \in V \text{ and } u(t) \in U \tag{3}$$

where $V \subset \mathbb{R}^K$ is convex and compact and $U \subset \mathbb{R}^N$ such that $U = \left\{ u \in \mathbb{R}^N ; a_i \le u_i \le 0, i = 1, 2, ..., N \right\}$ for fixed a_i

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Moreover, $w \in \mathcal{V}$ and $u \in \mathcal{U}$ where

$$\mathcal{V} = \left\{ w \in L^2 \left([t_0, t_1], \mathbb{R}^K \right); w(t) \in V \right\},$$

$$\mathcal{U} = \left\{ u \in L^2 \left([t_0, t_1], \mathbb{R}^N \right); u(t) \in U \right\}$$

are sets of admissible controls.

- $m : [t_0, t_1] \to \mathbb{R}_+$ is a function satisfying

$$m(t) = m(t_0) + \sum_{i=1}^{N} \int_{t_0}^{t} u_i(\tau)\, d\tau,$$

$$m(t_0) + (t_1 - t_0) \sum_{i=1}^{N} a_i \geq \underline{m} > 0 \quad (4)$$

and $m(t) \geq \underline{m} > 0$ on $[t_0, t_1]$ with a fixed $\underline{m}$.

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We also assume:

(A1) a matrix-valued function $G : [t_0, t_1] \times \mathbb{R}^n \longrightarrow \mathbb{R}^{n\times K}$ is continuous and there is a function $P : [t_0, t_1] \times \mathbb{R}^n \times \mathbb{R}^K \longrightarrow \mathbb{R}$ such that $P_x(t, x, w) = G(t, x) w$ and

$$P(t, x, w) \geq -\alpha_2 |x|^2 - \alpha_1 |x| - \alpha_0$$

for some constant numbers α_i, $i = 0, 1, 2$, where $\alpha_2 \leq \frac{m}{2}\left(\frac{\pi}{t_1 - t_0}\right)^2$, $t \in [t_0, t_1]$, $x \in \mathbb{R}^n$, $w \in V$.

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The functional of action for the system (1) has the form:

$$Q(x,w,u)=\frac{1}{2}\int_{t_0}^{t_1}|\dot{x}(t)|^2\,dt+$$
$$\int_{t_0}^{t_1}\frac{1}{m(t)}\left(P(t,x(t),w(t))+(C(t)u(t),x(t))\right)dt. \quad (5)$$

The functional Q is well-defined on the space $H^1([t_0,t_1],\mathbb{R}^n)$.

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Theorem 1. (On the existence of solutions)

If assumptions $(A1)$-$(A2)$ are satisfied, then for any admissible control $(w,u)\in\mathcal{V}\times\mathcal{U}$ there exist at least one trajectory for (1) satisfying boundary conditions (2) .

Theorem 2. (On the uniqueness of solutions)

If assumptions $(A1)$-$(A2)$ are satisfied and the function P is convex with respect to x, then for any $(w,u)\in\mathcal{V}\times\mathcal{U}$ system (1) possesses exactly one solution satisfying conditions (2).

One can relax the convexity assumption requiring only that the function $\beta\left|x\right|^2+P\left(t,x,w\right)$ is convex with respect to x for an arbitrary $t\in[t_0,t_1]$, $w\in V$ and some $\beta<\frac{1}{2}\left(\frac{\pi}{t_1-t_0}\right)^2$.

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Let

$\{(w_s, u_s)\}_{s=0}^{\infty}$ denote an arbitrary sequence of admissible controls convergent to (w_0, u_0) in an appropriate topology.

$X_s = X_{(w_s, u_s)}$ denote the set of all trajectories of the system (1) – (2) corresponding to a control (w_s, u_s).

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Theorem 3. (On the continuous dependence)

If for the control problem (1) – (4) the assumptions of Theorem 2 are satisfied, then for an arbitrary control (w_s, u_s) there exists the unique trajectory x_s for $s = 0, 1, \ldots$ and x_s converges to x_0 in $H^1([t_0, t_1], \mathbb{R}^n)$, if (w_s, u_s) tends to (w_0, u_0) in the weak topology of $L^2([t_0, t_1], \mathbb{R}^K \times \mathbb{R}^N)$.

In other words, if the assumptions of Theorem 2 are satisfied, then the operator $T : (w, u) \longmapsto x_{(w,u)} \in H^1([t_0, t_1], \mathbb{R}^n)$ is continuous with respect to the weak topology in the set of controls and the strong topology in the set of trajectories.

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Theorem 4. (On the semicontinuous dependence)

If for the control problem (1) – (4) the assumptions of Theorem 1 are satisfied, then

(a) sets $X_{(w,u)}$ are commonly bounded, i.e. there exist a ball $B(0,\rho)$ such that $X_{(w,u)} \subset B(0,\rho)$ for any $(w,u) \in \mathcal{V} \times \mathcal{U}$,

(b) $Lim \sup X_s \neq \emptyset$ and $Lim \sup X_s \subset X_0$, if a sequence (w_s, u_s) tends to (w_0, u_0) in the weak topology of $L^2([t_0, t_1], \mathbb{R}^K \times \mathbb{R}^N)$.

Definition

$Lim \sup X_s$ denotes the upper limit of a sequence $\{X_s\}_{s=1}^{\infty}$ which is a set of all cluster points of sequences $\{x_s\}_{s=1}^{\infty}$ such that $x_s \in X_s$ for $s = 1, 2, \ldots$.

Assertions (a), (b) mean that the multifunction $(w,u) \longmapsto X_{(w,u)}$ is upper semicontinuous with respect to the weak topology in the set of controls and the strong topology in the set of trajectories.

When for any $(w,u) \in \mathcal{V} \times \mathcal{U}$ the set $X_{(w,u)}$ is a singleton, i.e. $X_{(w,u)} = \{x_{(w,u)}\}$, the upper semicontinuity of the multifunction $(w,u) \longmapsto X_{(w,u)}$ can be reduced to the continuity of the operator $T : (w,u) \longmapsto x_{(w,u)}$.

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Next we shall consider the control problem (1) – (4) with the cost functional

$$I(x,w,u)=\int_{t_0}^{t_1}\Phi\left(t,x(t),\dot{x}(t),w(t),u(t)\right)dt. \tag{6}$$

(A3) an integrand $\Phi:[t_0,t_1]\times\mathbb{R}^n\times\mathbb{R}^n\times V\times U\to\mathbb{R}$ is continuous with respect to $(t,x,\dot{x},w,u)$, convex with respect to the controls (w,u) and for each $L>0$ there are a $\beta_L>0$ and a $\gamma_L>0$ such that

$$\left|\Phi\left(t,x,\dot{x},w,u\right)\right|\leq\beta_L\left|\dot{x}\right|^2+\gamma_L$$

for any $x\in\mathbb{R}^n$, $|x|<L$, $t\in[t_0,t_1]$, $w\in V$ and $u\in U$.

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for any $x\in\mathbb{R}^n$, $|x|<L$, $t\in[t_0,t_1]$, $w\in V$ and $u\in U$.

Theorem 5.

Suppose that the cost functional (6) satisfies assumption $(A3)$ and for the control problem $(1)-(4)$ the assumptions of Theorem 2 are fulfilled. Then for an arbitrary control (w,u) there exists the unique trajectory $x_{(w,u)}$ of the problem $(1)-(2)$ and in the set of all admissible processes $\left((w,u), x_{(w,u)}\right)$ there exists a process $\left((w^*,u^*), x_{(w^*,u^*)}\right)$ that minimizes the cost functional (6) and we call it optimal.

An analogous theorem one can prove when the set of trajectories $X_{(w,u)}$ corresponding to the control (w,u) contains many elements.

Theorem 5.

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An analogous theorem one can prove when the set of trajectories $X_{(w,u)}$ corresponding to the control (w, u) contains many elements.

In mechanics and the theory of decay of elementary particles the essential role plays the following equation

$$m(t)\,\dot{v}(t) = \sum_{i=1}^{N} (v_i(t) - v(t))\,\dot{m}_i(t) + f^{ext}(t). \tag{7}$$

The equation (7) is so called the Meščerskii rocket equation and describes a motion of some object with variable mass $m = m(t)$ and velocity $v = v(t)$, such as a rocket, an airplane and the like. This object is powered by N engines that emit gases with velocities $v_i = v_i(t)$, $i = 1, 2, ..., N$ which is a consequence of fuel combustion with the speed $\dot{m}_i(t)$. In equation (7), $f^{ext} = f^{ext}(t)$ denotes all external forces that stimulate the motion for example the gravity force and the force exerted by controls.

If we assume that

- only one mass is emitted $(N = 1)$,
- the external force is negligible $\left(f^{ext} = 0\right)$,
- a relative velocity of the emitted mass is constant $c = v_1(t) - v(t) < 0$ for $t \in [t_0, t_1]$,

then the equation (7) reduces to the well-known Tsiolkovskii equation

$$m(t)\,\dot{v}(t) = c\dot{m}(t). \tag{8}$$

Integrating equation (8), we obtain the following recipe for the velocity

$$v(t) = v(t_0) + c \ln \frac{m(t)}{m(t_0)}$$

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Information about the Meščerskii equation and the Tsiolkovskii equation can be found, among others, in:

- A.A. Kosmodemianskii, *The course of theoretical mechanics II*, Moscow (in Russian); 1966.
- J.L. Meriam and L.G. Kraige, *Engineering mechanics, Dynamics*, 5th edition, John Wiley & Sons; 2002.
- I.V. Meščerskii, *Works on mechanics of the variable mass*, Moscow (in Russian); 1962.
- M. Pardy, The rocket equation for decays of elementary particles, arXiv:hep-ph/0608161v1, 2008.
- J. Peraire, Variable mass systems: the rocket equation, MIT OpenSourceWare, Massachusetts Institute of Technology, Available online, 2004.

Our equation is of Meščerskii type if we impose the following assumptions:

(A) the external force f^{ext} depends on the location of the object

$$x(t) = \bar{x}_0 + \int_{t_0}^{t} v(\tau)\, d\tau$$

and on the controls exerting the force $w(t)$ thus attaining the form

$$f^{ext}(t) = f\left(t, \bar{x}_0 + \int_{t_0}^{t} v(\tau)\, d\tau\right) w(t).$$

(B) we can control the speed of fuel combustion, i.e.

$$u_i(t) = \dot{m}_i(t) \in [a_i, 0].$$

(C) the relative velocity of the emitted mass

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Under the above assumptions the Meščerskii equation admits the integro-differential form

$$m(t)\,\dot{v}(t) = \sum_{i=1}^{N} c_i(t)\,u_i(t) + f\left(t, \overline{x}_0 + \int_{t_0}^{t} v(\tau)\,d\tau\right) w(t)\,. \tag{9}$$

After the following substitutions

$$x(t) = \overline{x}_0 + \int_{t_0}^{t} v(\tau)\,d\tau,$$
$$\dot{x}(t) = v(t)\,,$$
$$\sum_{i=1}^{N} c_i(t)\,u_i(t) = C(t)\,u(t)\,,$$
$$f(t, x(t)) = G(t, x(t))$$

we finally obtain from the Meščerskii equation the equation (1).

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After the following substitutions

$$\begin{aligned} x(t) &= \overline{x}_0 + \int_{t_0}^{t} v(\tau)\,d\tau, \\ \dot{x}(t) &= v(t), \\ \sum_{i=1}^{N} c_i(t)\,u_i(t) &= C(t)\,u(t), \\ f(t, x(t)) &= G(t, x(t)) \end{aligned}$$

we finally obtain from the Meščerskii equation the equation (1).

The results of the paper may be extended to the 2D continuous system of the form

$$z_{tx}(t,x) + \frac{a}{m_0 - \int\limits_0^t u(\tau)\,d\tau} z_t(t,x) + b z_x(t,x) = 0 \tag{10}$$

with the boundary conditions

$$z(0,x) = \varphi(x), \;\; z(t,0) = \psi(t), \;\; \varphi(0) = \psi(0) \tag{11}$$

and the integral condition

$$m_0 - \int\limits_0^t u(\tau)\,d\tau \geq m_1 > 0$$

for $(t,x) \in [0,t_1] \times [0,x_1]$.

System (10) – (11) describe the process of filtration of a mixture of liquids (for example a mixture of water and some liquid with toxic substances) by passing it through the filter made in the form of a vertical pipe.
For details see for example:

A.N. Tikhonov and A.A. Samarskii, *Equations of Mathematical Physics*, Dover, New York; 1990.

In system (10) – (11) :

- $z = z(t, x)$ stands for the toxic liquid concentration at a moment t and at a distance x from the inlet of the pipe,
- $u(t)$ denotes the speed of the flow of liquids from the interval $[0, u_1]$,
- constants a and b are some physical quantities,
- m_1 is the mass of the filter,
- $m_0 - m_1$ is the mass of the mixture we want to filter.

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System (10) – (11) is a counterpart of the discrete model of Fornasini-Marchesini type that was examined in many paper with various cost functionals.

D. Idczak, K. Kibalczyc and S. Walczak, On an optimization problem with cost of rapid variation of control, *The Journal of the Australian Mathematical Society, Series B,* Vol. 36, No. 1, 1994, pp.117-131.

Numerical algorithm for finding optimal solutions to the discrete version of the process of filtration was presented in

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Thank you for your attention!