Controlled and Conditioned Invariance with Stability for Two-Dimensional Systems

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Outline

- Motivations
- 1-D Controlled and Conditioned Invariance
- Fornasini-Marchesini models
- 2-D Controlled and Conditioned Invariance
- Construction of (local) state feedback and output injection

Motivations

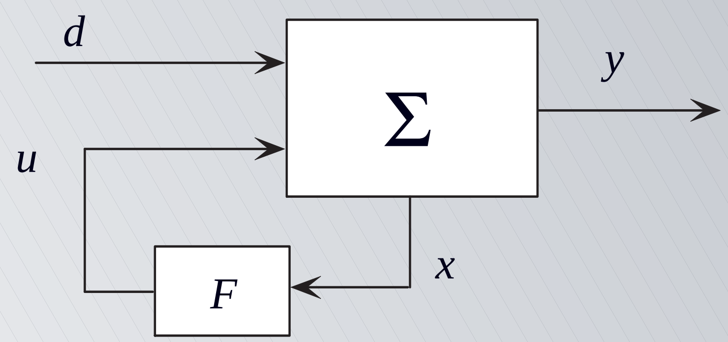
Where does the need to develop
geometric methods for 2-D systems originate?

- Insight into many system-theoretic properties of linear systems (invariant zeros, left and right invertibility, relative degree, etc.)
- Simple solutions to problems that are very hard to solve otherwise (disturbance decoupling, full information control, unknown-input observation, singular/cheap LQ problems, non-interaction, etc.)

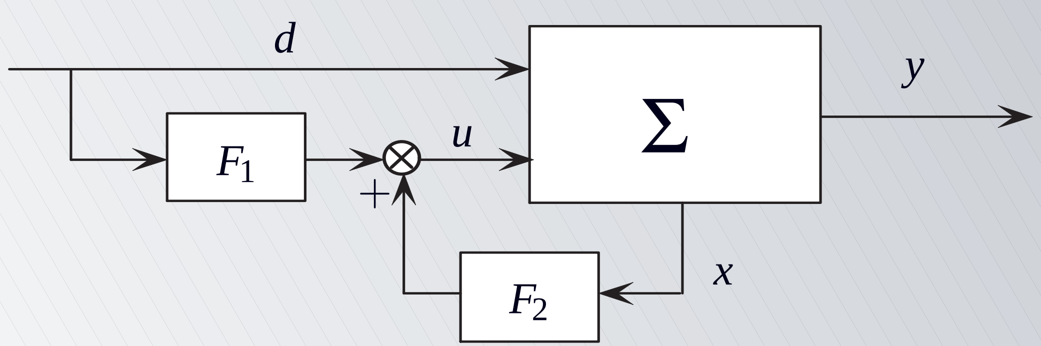
Motivation: Disturbance Decoupling Problems

Controlled Invariance is the key tool to solve *Disturbance Decoupling Problems*:

- Decoupling of non-measured disturbances:



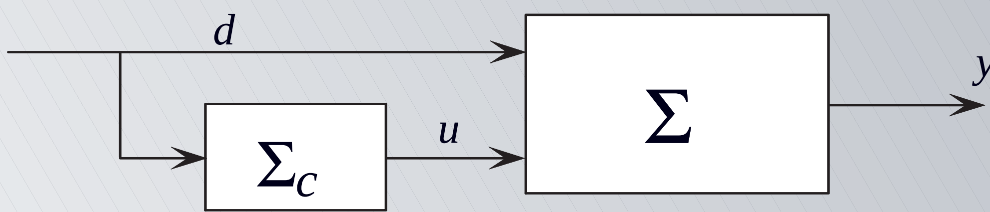
- Decoupling of measurable disturbances:



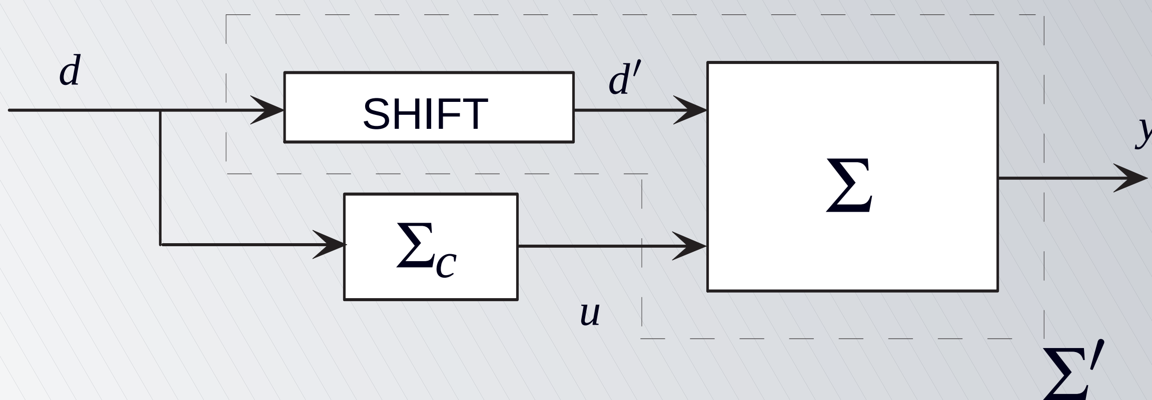
Motivation: Disturbance Decoupling Problems

Controlled Invariance is the key tool to solve *Disturbance Decoupling Problems*:

- Full Information for measurable disturbaces:



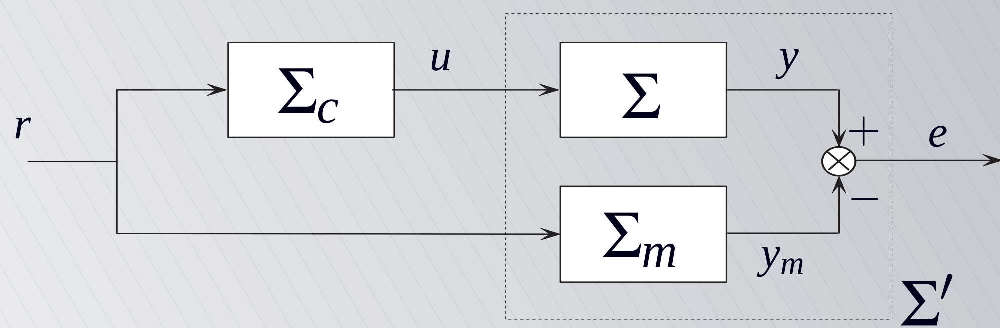
- Full Information for *previewed* disturbances:



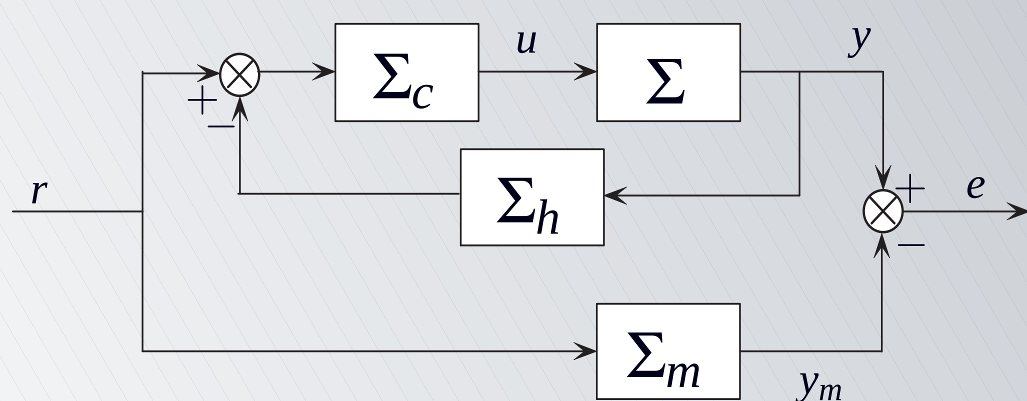
Motivation: Tracking Problems

Controlled Invariance is the key tool to solve *Model Matching Problems*:

- Model Matching (Feedforward Scheme):



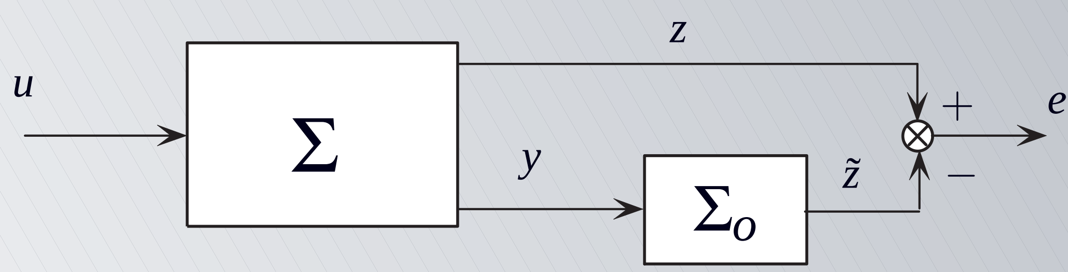
- Model Matching (Feedback Scheme):



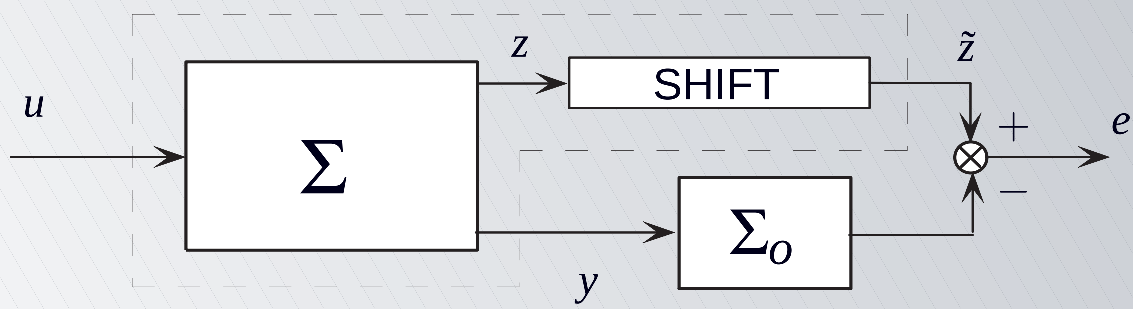
Motivation: Unknown-Input Observation Problems

Conditioned Invariance is the key tool to solve *Unknown-Input Observation Problems*:

- Unknown-Input Observer:



- Fixed-Lag Smoothing:



1-D Controlled Invariants (Basile & Marro, 1969)

For 1-D system (A, B, C, D):

- Controlled Invariant Subspaces:

$$\boxed{A\mathscr{V} \subseteq \mathscr{V} + \operatorname{im} B}$$

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- Output-Nulling Subspaces:

$$\begin{bmatrix} A \\ C \end{bmatrix} \mathscr{V} \subseteq (\mathscr{V} \times \{0\}) + \operatorname{im} \begin{bmatrix} B \\ D \end{bmatrix}$$

1-D Conditioned Invariants (Basile & Marro, 1969)

For 1-D systems (A, B, C, D):

- Conditioned Invariant Subspaces:

$$\boxed{A(\mathscr{S} \cap \ker C) \subseteq \mathscr{S}}$$

1-D Conditioned Invariants (Basile & Marro, 1969).

For 1-D systems (A, B, C, D):

- Conditioned Invariant Subspaces:

$$\boxed{A(\mathscr{S} \cap \ker C) \subseteq \mathscr{S}}$$

- Input-Containing Subspaces:

$$\boxed{\begin{bmatrix} A & B \end{bmatrix} (\mathscr{S} \times \mathbb{R}^m \cap \ker \begin{bmatrix} C & D \end{bmatrix}) \subseteq \mathscr{S}}$$

Duality

The dual of the quadruple (A, B, C, D) is $(A^\top, C^\top, B^\top, D^\top)$.

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$$\left\{ \begin{aligned} x_{k+1} &= Ax_k + Bu_k \\ y_k &= Cx_k + Du_k \end{aligned} \right. \qquad \longleftrightarrow \qquad \left\{ \begin{aligned} \tilde{x}_{k+1} &= A^\top \tilde{x}_k + C^\top \tilde{u}_k \\ \tilde{y}_k &= B^\top \tilde{x}_k + D^\top \tilde{u}_k \end{aligned} \right.$$

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$$\Sigma = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \qquad \longleftrightarrow \qquad \Sigma^\top = \begin{bmatrix} A^\top & C^\top \\ B^\top & D^\top \end{bmatrix}$$

Duality

- Controlled and Conditioned Invariants are dual concepts:

  $\mathscr{V}$ is Controlled Invariant for Σ iff $\mathscr{V}^{\perp}$ is Conditioned Invariant for $\Sigma^{\top}$.

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  $\mathscr{V}$ is Controlled Invariant for Σ iff $\mathscr{V}^{\perp}$ is Conditioned Invariant for $\Sigma^{\top}$.

- Output-Nulling and Input-Containing Subspaces are dual concepts:

  $\mathscr{V}$ is Output-Nulling for Σ iff $\mathscr{V}^{\perp}$ is Input-Containing for $\Sigma^{\top}$.

Fornasini-Marchesini Models

The Kurek form of a 2-D Fornasini-Marchesini model is

$$\left\{\begin{aligned} x_{i+1,j+1} &= A_0 x_{i,j} + A_1 x_{i+1,j} + A_2 x_{i,j+1} \\ &\quad + B_0 u_{i,j} + B_1 u_{i+1,j} + B_2 u_{i,j+1} \\ y_{i,j} &= C x_{i,j} + D u_{i,j} \end{aligned}\right.$$

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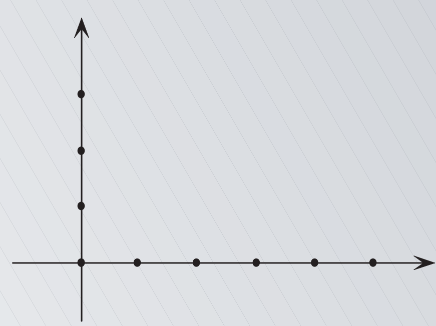
Boundary Conditions:



$$\mathfrak{B} = \Big(\mathbb{N} \times \{0\}\Big) \cup \Big(\{0\} \times \mathbb{N}\Big)$$

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2-D Controlled Invariant Subspaces:

$$\begin{bmatrix} A_0 \\ A_1 \\ A_2 \end{bmatrix} \mathscr{V} \subseteq (\mathscr{V} \times \mathscr{V} \times \mathscr{V}) + \mathrm{im} \begin{bmatrix} B_0 \\ B_1 \\ B_2 \end{bmatrix},$$

(Conte & Perdon, 1988).

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Notation: Given matrices M_0, M_1 , M_2:

$$M_V = \begin{bmatrix} M_0 \\ M_1 \\ M_2 \end{bmatrix}, \quad M_H = \begin{bmatrix} M_0 & M_1 & M_2 \end{bmatrix}, \quad M_D = \begin{bmatrix} M & O & O \\ O & M & O \\ O & O & M \end{bmatrix}.$$

Hence, $\mathscr{V}$ is controlled invariant if $A_V \mathscr{V} \subseteq \mathscr{V}_D + \operatorname{im} B_V$, where $\mathscr{V}_D \triangleq \mathscr{V} \times \mathscr{V} \times \mathscr{V}$.

Synopsis for Controlled Invariance

- A 1-D Controlled Invariant is a subspace $\mathscr{V}$ s.t.

$$\boxed{A\mathscr{V} \subseteq \mathscr{V} + \operatorname{im} B}$$

- A 2-D Controlled Invariant is a subspace $\mathscr{V}$ s.t.

$$\boxed{A_V\mathscr{V} \subseteq \mathscr{V}_D + \operatorname{im} B_V}$$

- A 1-D Output-Nulling subspace $\mathscr{V}$ is s.t.

$$\boxed{\begin{bmatrix} A \\ C \end{bmatrix} \mathscr{V} \subseteq (\mathscr{V} \times \{0\}) + \operatorname{im} \begin{bmatrix} B \\ D \end{bmatrix}}$$

- A 2-D Output-Nulling subspace $\mathscr{V}$ is s.t.

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Duals of Fornasini-Marchesini Models

For the special FM models

$$\begin{cases} x_{i+1,j+1} = A_0 x_{i,j} + A_1 x_{i+1,j} + A_2 x_{i,j+1} + B u_{i,j} \\ y_{i,j} = C x_{i,j} + D u_{i,j} \end{cases}$$

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and

$$\begin{cases} x_{i+1,j+1} = A_1 x_{i+1,j} + A_2 x_{i,j+1} + B_1 u_{i+1,j} + B_2 u_{i,j+1} \\ y_{i,j} = C_1 x_{i+1,j} + C_2 x_{i,j+1} + D_1 u_{i+1,j} + D_2 u_{i,j+1} \end{cases}$$

a dual can be easily defined (see Karamancioğlu and Lewis, 1992).

Synopsis for Conditioned Invariance

- A 1-D Conditioned Invariant is a subspace $\mathscr{S}$ s.t.

$$\boxed{A(\mathscr{S} \cap \ker C) \subseteq \mathscr{S}}$$

- A 2-D Conditioned Invariant is a subspace $\mathscr{S}$ s.t.

$$\boxed{A_H(\mathscr{S}_D \cap \ker C_D) \subseteq \mathscr{S}}$$

- A 1-D Input-Containing subspace $\mathscr{S}$ is s.t.

$$\boxed{[A \quad B](\mathscr{S} \times \mathbb{R}^m \cap \ker [C \quad D]) \subseteq \mathscr{S}}$$

- A 2-D Input-Containing subspace $\mathscr{S}$ is s.t.

$$\boxed{[A_H \quad B_H](\mathscr{S}_D \times \mathbb{R}^{3m} \cap \ker [C_D \quad D_D]) \subseteq \mathscr{S}}$$

Controlled Invariant Subspaces: Interpretation

The subspace $\mathscr{V} \subseteq \mathbb{R}^n$ is controlled invariant if

$$\boxed{A_V \mathscr{V} \subseteq \mathscr{V}_D + \operatorname{im} B_V}$$

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A controlled invariant $\mathscr{V}$ is such that the FM model admits a solution in $x_{i,j} \in \mathscr{V}$ for any $\mathscr{V}$-valued boundary condition:

$$\boxed{x_{i,j} \in \mathscr{V} \;\; \forall (i,j) \in \mathfrak{B} \implies \exists u_{i,j} : \; x_{i,j} \in \mathscr{V} \;\; \forall i,j \geq 0}$$

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Such control $u_{i,j}$ can *always* be expressed as

$$\boxed{\boxed{u_{i,j} = F\, x_{i,j}}}$$

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The subspace $\mathscr{V} \subseteq \mathbb{R}^n$ is output-nulling if

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$$\boxed{x_{i,j} \in \mathscr{V}\ \forall (i,j) \in \mathfrak{B} \implies \exists u_{i,j} : x_{i,j} \in \mathscr{V} \textbf{ and } y_{i,j} = 0\ \forall i,j \geq 0}$$

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Controlled Invariance and Local State Feedback

Plugging $u_{i,j} = F\,x_{i,j}$ into the FM model and after a change of coordinates with $T = [\,T_1 \;\; T_2\,]$ with $\operatorname{im} T_1 = \mathscr{V}$, we get

$$\begin{bmatrix} x'_{i+1,j+1} \\ x''_{i+1,j+1} \end{bmatrix} = \begin{bmatrix} A_0^{11}(F) & A_0^{12}(F) \\ 0 & A_0^{22}(F) \end{bmatrix} \begin{bmatrix} x'_{i,j} \\ x''_{i,j} \end{bmatrix} + \begin{bmatrix} A_1^{11}(F) & A_1^{12}(F) \\ 0 & A_1^{22}(F) \end{bmatrix} \begin{bmatrix} x'_{i+1,j} \\ x''_{i+1,j} \end{bmatrix} + \begin{bmatrix} A_2^{11}(F) & A_2^{12}(F) \\ 0 & A_2^{22}(F) \end{bmatrix} \begin{bmatrix} x'_{i,j+1} \\ x''_{i,j+1} \end{bmatrix}$$

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- $x'_{i,j}$ is the *internal* component of $x_{i,j}$ on $\mathscr{V}$;
- $x''_{i,j}$ is the *external* component of $x_{i,j}$ w.r.t. $\mathscr{V}$.

Problem: Find a friend of $\mathscr{V}$ such that the internal and external components of the local state are both stable.

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- If $\exists F$ such that $\left(A_0^{11}(F), A_1^{11}(F), A_1^{11}(F)\right)$ is asympt. stable, $\mathscr{V}$ is said to be *internally stabilisable*;
- If $\exists F$ such that $\left(A_0^{22}(F), A_1^{22}(F), A_1^{22}(F)\right)$ is asympt. stable, $\mathscr{V}$ is said to be *externally stabilisable*.

Controlled Invariance and Local State Feedback

Consider again

$$\begin{bmatrix} x'_{i+1,j+1} \\ x''_{i+1,j+1} \end{bmatrix} = \begin{bmatrix} A_0^{11}(F) & A_0^{12}(F) \\ 0 & A_0^{22}(F) \end{bmatrix} \begin{bmatrix} x'_{i,j} \\ x''_{i,j} \end{bmatrix} + \begin{bmatrix} A_1^{11}(F) & A_1^{12}(F) \\ 0 & A_1^{22}(F) \end{bmatrix} \begin{bmatrix} x'_{i+1,j} \\ x''_{i+1,j} \end{bmatrix} + \begin{bmatrix} A_2^{11}(F) & A_2^{12}(F) \\ 0 & A_2^{22}(F) \end{bmatrix} \begin{bmatrix} x'_{i,j+1} \\ x''_{i,j+1} \end{bmatrix}$$

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Given $\mathscr{V}$, there are many *friends* F. How to select those F for which

- $\left(A_0^{11}(F), A_1^{11}(F), A_2^{11}(F)\right)$ is asympt. stable?
- $\left(A_0^{22}(F), A_1^{22}(F), A_2^{22}(F)\right)$ is asympt. stable?

Controlled Invariance and Local State Feedback

Let V be a basis of $\mathscr{V}$. The following are equivalent:

- The subspace $\mathscr{V}$ is controlled invariant:

$$A_V \mathscr{V} \subseteq \mathscr{V}_D + \operatorname{im} B_V$$

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- $\exists X, \Omega$ such that

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- $\exists F, X$ such that

$$(A_V + B_V F) V = V_D X$$

- $\mathscr{V}$ is $(A_i + B_i F)$-invariant ($i \in \{0, 1, 2\}$).

In order to find F:

a) Solve $A_V V = V_D X + B_V \Omega$:

$$\begin{bmatrix} X \\ \Omega \end{bmatrix} = [V_D \;\; B_V]^{\dagger} A_V V + H_1 K_1 \quad \text{where} \;\; \mathrm{im}\, H_1 = \ker [V_D \;\; B_V]$$

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b) Let F be such that $\Omega = -F V$:

$$F = -\Omega (V^{\top} V)^{-1} V^{\top} + K_2 H_2, \quad \text{where} \;\; \ker H_2 = \mathrm{im}\, V$$

$$\Downarrow$$

Two degrees of freedom in the choice of F: matrices K_1 and K_2.

Controlled Invariance and Local State Feedback

Consider again

$$\begin{bmatrix} x'_{i+1,j+1} \\ x''_{i+1,j+1} \end{bmatrix} = \begin{bmatrix} A_0^{11}(K_1,K_2) & A_0^{12}(K_1,K_2) \\ 0 & A_0^{22}(K_1,K_2) \end{bmatrix} \begin{bmatrix} x'_{i,j} \\ x''_{i,j} \end{bmatrix} + \begin{bmatrix} A_1^{11}(K_1,K_2) & A_1^{12}(K_1,K_2) \\ 0 & A_1^{22}(K_1,K_2) \end{bmatrix} \begin{bmatrix} x'_{i+1,j} \\ x''_{i+1,j} \end{bmatrix} + \begin{bmatrix} A_2^{11}(K_1,K_2) & A_2^{12}(K_1,K_2) \\ 0 & A_2^{22}(K_1,K_2) \end{bmatrix} \begin{bmatrix} x'_{i,j+1} \\ x''_{i,j+1} \end{bmatrix}$$

Controlled Invariance and Local State Feedback

Consider again

$$\begin{bmatrix} x'_{i+1,j+1} \\ x''_{i+1,j+1} \end{bmatrix} = \begin{bmatrix} A_0^{11}(K_1,K_2) & A_0^{12}(K_1,K_2) \\ 0 & A_0^{22}(K_1,K_2) \end{bmatrix} \begin{bmatrix} x'_{i,j} \\ x''_{i,j} \end{bmatrix} + \begin{bmatrix} A_1^{11}(K_1,K_2) & A_1^{12}(K_1,K_2) \\ 0 & A_1^{22}(K_1,K_2) \end{bmatrix} \begin{bmatrix} x'_{i+1,j} \\ x''_{i+1,j} \end{bmatrix}$$
$$+ \begin{bmatrix} A_2^{11}(K_1,K_2) & A_2^{12}(K_1,K_2) \\ 0 & A_2^{22}(K_1,K_2) \end{bmatrix} \begin{bmatrix} x'_{i,j+1} \\ x''_{i,j+1} \end{bmatrix}$$

- matrices $A_i^{11}(K_1,K_2)$ do not depend on K_2
- matrices $A_i^{22}(K_1,K_2)$ do not depend on K_1

$$\Downarrow$$

Two independent procedures to find K_1 and K_2.

Input-Containing Subspaces: Interpretation

The subspace $\mathscr{S} \subseteq \mathbb{R}^n$ is input-containing if

$$\boxed{\begin{bmatrix} A_H & B_H \end{bmatrix} (\mathscr{S}_D \times \mathbb{R}^{3m} \cap \ker \begin{bmatrix} C_D & D_D \end{bmatrix}) \subseteq \mathscr{S}}$$

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An input-containing $\mathscr{S}$ is such that an observer for the FM ruled by

$$\begin{aligned}\omega_{i+1,j+1} \;=\;& K_0\,\omega_{i,j} + K_1\,\omega_{i+1,j} + K_2\,\omega_{i,j+1} + N_0\,u_{i,j} + N_1\,u_{i+1,j} \\ & + N_2\,u_{i,j+1} + L_0\,y_{i,j} + L_1\,y_{i+1,j} + L_2\,y_{i,j+1}\end{aligned}$$

exists such that

$$\boxed{\omega_{i,j} = x_{i,j}/\mathscr{S} \;\; \forall\,(i,j) \in \mathfrak{B} \implies \omega_{i,j} = x_{i,j}/\mathscr{S} \;\; \forall\, i,j \geq 0}$$

Input-Containing Subspaces and Output Injection

It can be shown that input-containing subspaces are linked to the existence of *output-injection* matrices G such that

$$\begin{bmatrix} A_H + GC_D & B_H + GD_D \end{bmatrix} \left(\mathscr{S}_D \times \mathbb{R}^{3m} \right) \subseteq \mathscr{S}$$

The notion of *external stabilisability* of an input-containing subspace can lead to a definition of *detectability subspace*.

Given a detectability subspace $\mathscr{S}$, we can construct an observer such that $\omega_{i,j}$ converges to $x_{i,j}/\mathscr{S}$ as (i,j) evolves away from $\mathfrak{B}$.

Concluding remarks and future works

Concluding remarks:

- Notions of 2-D controlled and conditioned invariance with stabilisability properties;
- Simple (constructive) solutions to disturbance decoupling and unknown-input observation problems.

Future work:

- Reachability subspaces and invariant zeros;
- Geometric solution to singular LQ problems.

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